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Adaptive Almost Disturbance Decoupling for a Class of Uncertain Nonlinear Systems

Cui-Hua Zhang[1], Zhuo Wang[2], Bing Cai[3], Zong-Yao Sun[1], Qing-Quan Tan[4]

1. Institute of Automation, Qufu Normal University, Qufu, Shandong Province, 273165, China

2. School of Instrumentation Science and Optoelectronics Engineering, Beihang University, Beijing, 100191, China
E-mail: zhuowang@buaa.edu.cn

3. School of Electrical Information and Automation ,Qufu Normal University, Qufu, Shandong Province, 273165, China

4. Earthquake Administration of Beijing Municipality, Beijing, 100080, P. R. China

Abstract: This paper investigates the problem of adaptive almost disturbance decoupling for a class of generalized high-order uncertain nonlinear systems. The control strategy is on the basis of continuous domination and delicate adaptive technique and the adaptive state-feedback controller is one-dimensional irrespective of the number of unknown parameters. An appropriate nonlinear function and transformation skill are introduced to mitigate the effects of external disturbances. As an application, an example is provided to illustrate the correctness of the theoretical results.

Key Words: Adaptive almost disturbance decoupling, High-order uncertain nonlinear systems, Continuous domination

1 Introduction

It is well known that high-order nonlinear systems have uncontrollable linearization around the origin, so its stabilization has been viewed as one of the most challenging issues. Fortunately, many results have been obtained with the help of adding a power integrator method and homogeneous domination idea, such as [1–14], to name just a few.

On the other hand, practical control systems are always corrupted by various types of unknown disturbances, and one topic in control design is to attenuate their influence on the output as much as possible, since it is hard to realize exact disturbance decoupling. What is worse, uncertainties also have a potential tendency to deteriorate system performance or even destabilize control systems, so their effects have to be taken into consideration. Discarding parameter uncertainties, the topic has been solved partly, see [15–18] and references therein. Specifically, in light of internal stability and a feedback domination design, [15] and [16] solved it for linear systems and nonlinear systems by providing necessary and sufficient geometric conditions, respectively. [17] discussed the problems of input-to-state stability with respect to disturbance inputs and almost disturbance decoupling output tracking for strict feedback nonlinear systems, and [18] presented an approach to output feedback stabilization with L_2 gain disturbance attenuation in the presence of zero dynamics. Furthermore, [19] made an interesting exploration for a class of nonlinearly parameterized systems, and [20] permitted the existence of more uncertainties including unknown parameters and unmeasurable states. In comparison, there is little progress on almost disturbance decoupling of high-order nonlinear systems, because it is really difficult to construct state observer and Lyapunov function satisfying assumptions of internal stability in a complex environment. Fortunately, the paper [21] formulated a well posed almost disturbance decoupling problem for the first time, and illustrated how to utilize the adding a power integrator method to construct a smooth state-feedback control law, while there are no uncertainties in the systems. Therefore, one may propose a natural and interesting question: *How large uncertainties will be allowed to construct a feedback controller for high-order nonlinear systems in the presence of external disturbances?* It is worth emphasizing that the affirmative solution to above question is a troublesome task, which can be seen from two aspects. (i) The first difficulty is the identification of uncertainties. Undiscovered parts of the systems can be composed of unmeasurable states, unknown parameters and unclear structures, except for possible disturbance. This paper puts a foothold in dealing with unknown parameters. In order to suppress uncertainties simultaneously, we introduce an appropriate nonlinear function, and use transformation skill combined with adaptive technique to alleviate their effects. (ii) The second difficulty is the simplification of the controller. Its remarkable feature is that the order of dynamic compensator is equal to one, which simplifies the procedure of control design and stability analysis of the closed-loop systems. Under relaxed conditions, the designed adaptive controller guarantees stabilization properties when external disturbance is absent, and attenuates the influence of the disturbance on the output with an arbitrary degree of accuracy in terms of L_2-L_{2p} gain.

We adopt the following notations throughout this paper. $\mathbb{R}^+$ denotes the set of all non-negative real numbers, and $\mathbb{R}^n$ denotes Euclidean space with dimension n. $R_{odd}^{\geq 1} \triangleq \{\frac{p}{q} |\ p$ and q are positive odd integers, and $p \geq q\}$. For a real vector $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, $\bar{x}_i \triangleq [x_1, \dots, x_i]^T \in \mathbb{R}^i$, $i = 1, \dots, n$, especially $\bar{x}_n = x$, and the norm $\|x\|$ of $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. The space L_p

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with $1 \leq p \leq \infty$ is defined as the set of all piecewise continuous functions $x : [0, \infty) \to \mathbb{R}^n$ such that $\|x\|_{L_p} = (\int_0^\infty \|x(t)\|^p \mathrm{d}t)^{1/p} < \infty$, $\|x\|_{L_\infty} = \sup_{t\geq 0} \|x(t)\| < \infty$. For a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}^+$, it is positive definite if $V(x) \geq 0$ and $V(x) = 0$ if and only if $x = 0$; it is radially unbounded if $V(x) \to \infty, \|x\| \to \infty$. The arguments of functions are sometimes simplified, a function $f(x(t))$ can be denoted by $f(x)$, $f(\cdot)$ or f.

2 Problem Formulation

We consider the following uncertain nonlinear systems

$$\begin{cases} \dot{x}_i = d_i(t,x,u,\theta)x_{i+1}^{p_i} + f_i(t,x,u,\theta) + g_i(t,x,u,\theta)\omega, \\ \dot{x}_n = d_n(t,x,u,\theta)u^{p_n} + f_n(t,x,u,\theta) + g_n(t,x,u,\theta)\omega, \\ y = h(x_1), \end{cases} \tag{1}$$

where $i = 1, \ldots, n-1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are system state, control input and system output, respectively. Initial condition is $x(0) = x_0$, and $u \triangleq x_{n+1}$. $\omega : \mathbb{R}^+ \to \mathbb{R}^s$ is a continuous time-varying disturbance signal satisfying $\omega \in L_2$, and $\theta \in \mathbb{R}^m$ represents an unknown parameter vector which can be time-invariable or time-varying. For each $i = 1, \ldots, n$, $p_i \in \mathbb{R}^{\geq 1}_{odd}$ is named by the power of the systems, and $f_i(\cdot)$, $g_i(\cdot)$ and $d_i(\cdot)$ are continuous nonlinear functions, while $h(x_1)$ is a continuously differentiable function with $h(0) = 0$.

The objective of this paper is given as follows:

Adaptive Almost Disturbance Decoupling(AADD): For system (1), find a continuous adaptive controller

$$\begin{cases} u(t) = u(x(t), \hat{\Theta}(t)),\ u(0, \hat{\Theta}(t)) = 0, \\ \dot{\hat{\Theta}}(t) = \tau(x(t), \hat{\Theta}(t)),\ \tau(0, \hat{\Theta}(t)) = 0, \end{cases} \tag{2}$$

where $\hat{\Theta}(t)$ is on-line estimate of unknown parameter Θ depending on θ, such that closed-loop systems composed of (1) and (2) satisfy the following features.

(i) When $\omega(t) = 0$, states of the closed-loop systems are globally uniformly bounded on the interval $[0, \infty)$, and $\lim_{t\to\infty} x(t) = 0$.

(ii) When $\omega(t) \in L_2$, for any pre-given small real number $\varepsilon > 0$, there holds $\int_0^t |y(s)|^{2p_1} \mathrm{d}s \leq \varepsilon^2 \int_0^t \|\omega(s)\|^2 \mathrm{d}s + \delta(x(0), \hat{\Theta}(0))$, $\forall t \in [0, \infty]$, where $\delta(\cdot)$ is nonnegative and rests with initial states of the closed-loop systems.

The following assumptions are needed.

Assumption 1. For each $i = 1, \ldots, n$, there is $0 < a_i\lambda_i(\bar{x}_i) \leq |d_i(\cdot)| \leq \mu_i(\bar{x}_{i+1}, \theta)$, where a_i is an unknown constant, λ_i is a positive smooth function, and μ_i is a continuous function.

Assumption 2. For each $i = 1, \ldots, n$, there exist nonnegative continuous functions $f_{il}(\bar{x}_i, \theta)$ with $f_{il}(0, \theta) = 0$, such that $|f_i(\cdot)| \leq \sum_{l=1}^{j_i} f_{il}(\bar{x}_i, \theta)|x_{i+1}|^{q_{il}}$, where j_i's are finite positive integers, and q_{il}'s are real numbers satisfying $0 \leq q_{i1} < q_{i2} < \cdots < q_{ij_i} < p_i$.

Assumption 3. For each $i = 1, \ldots, n$, there exists a nonnegative and continuously differentiable function $\varphi_i(\bar{x}_i, \theta)$ with $\varphi_i(0, \theta) = 0$, such that $\|g_i(\cdot)\| \leq \varphi_i(\bar{x}_i, \theta)$.

In what follows we explain the necessity of Assumptions 1-3 and exhibit how to enlarge the scope of the nonlinear systems through a remark.

Remark 1. Assumption 1 indicates that $d_i(\cdot)$ is strictly either positive or negative. Without loss of generality, we just consider the case of $d_i > 0$ in subsequent control design. Compared with assumptions in [2, 7], the paper relaxes upper bound μ_i of $|d_i|$ to a function of $x_1, \ldots, x_{i+1}, \theta$, except for the existence of unknown lower bound of $|d_i|$, hence more delicate manipulation technique should be introduced to achieve the desired control objective. Some complex deductions can change Assumption 2 into $|f_i| \leq \frac{d_i}{2}|x_{i+1}^{p_i}| + f_i^*(\bar{x}_i, \theta) \sum_{j=1}^{i} |x_j|$, but not vice versa, where f_i^* is a positive smooth function. This inequality is frequently used in the literature, such as [2, 4, 7, 12]. Assumption 3 is somewhat weaken than those in [20, 21], due to the coupling of ω and θ. □

Then, we list several technical lemmas which is used to construct the stabilizer in the control design.

Lemma 1.[1] For given $r \geq 0$ and every $x \in \mathbb{R}$, $y \in \mathbb{R}$, there holds $|x + y|^r \leq c_r(|x|^r + |y|^r)$, where $c_r = 2^{r-1}$ if $r \geq 1$, and $c_r = 1$ if $0 \leq r < 1$. Moreover, if r is a ratio of positive odd integers and $0 < r \leq 1$, there holds $|x^r - y^r| \leq 2^{1-r}|x - y|^r$.

Lemma 2.[1] For given positive real numbers m, n and a function $a(x, y)$, there holds $|a(x, y)x^m y^n| \leq c(x, y)|x|^{m+n} + \frac{n}{m+n}\left(\frac{m}{(m+n)c(x,y)}\right)^{\frac{m}{n}} |a(x, y)|^{\frac{m+n}{n}} |y|^{m+n}$, where $c(x, y) > 0$, for any $x \in \mathbb{R}, y \in \mathbb{R}$.

Lemma 3.[2] For a continuous function $f(x, y)$ with $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, there exist smooth functions $a(x) \geq 0, b(y) \geq 0, c(x) \geq 1, d(y) \geq 1$, such that $|f(x, y)| \leq a(x) + b(y), |f(x, y)| \leq c(x)d(y)$.

Lemma 4.[23] Consider a continuously differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$. Suppose $f, \dot{f} \in L_\infty$, and $f \in L_p$ for some $p \in [1, \infty)$, then $\lim\limits_{t\to\infty} f(t) = 0$.

Lemma 5. Let $f(x, y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function and $f(0, y) = 0$, there exists a positive smooth function $\tilde{f}(x, y)$ such that $f(x, y) \leq \tilde{f}(x, y) \sum_{i=1}^{n} |x_i|$.

Proof. The proof is omitted due to limited place.

3 Main Results

Theorem 1. If system (1) satisfies Assumptions 1-3, AADD problem is solvable by constructing one dimensional continuous adaptive controller.

Proof. The proof is based on an inductive argument which simultaneously constructs a Lyapunov function as well as a continuous adaptive state-feedback controller. It is divided into two parts.

Part I: the design of the controller

Step 0. As the initial step of control design, we introduce coordinate transformations as follows.

$$z_1 = x_1,\ z_k = x_k^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}},\ k=2,\ldots,n+1, \quad (3)$$

where $\alpha_i = -(\beta_i z_i)^{1/(p_1\cdots p_i)}$, $i = 1,\ldots,k-1$, and $u = \alpha_n$ which also implies $z_{n+1} = 0$. $\beta_1,\ldots,\beta_n$ are positive smooth functions to be specified later. Let $p_0 = 1$, $\alpha_0 = 0$. With the help of (3), one can find nonnegative smooth functions $\bar{\gamma}_i(\bar{x}_i,\hat{\Theta})$, $\bar{\varphi}_i(\bar{x}_i,\hat{\Theta})$, $\bar{\mu}_i(\bar{x}_{i+1})$, and an unknown constant $\bar{\Theta}_1 \geq 1$ such that for $i = 1,\ldots,n$,

$$\begin{cases} |f_i| \leq \frac{d_i}{2}|x_{i+1}^{p_i}| + \bar{\Theta}_1\bar{\gamma}_i \sum_{j=1}^{i} |z_j|^{\frac{1}{p_1\cdots p_{i-1}}}, \\ \varphi_i \leq \bar{\Theta}_1\bar{\varphi}_i \sum_{j=1}^{i} |z_j|^{\frac{1}{p_1\cdots p_{i-1}}},\ \ \mu_i \leq \bar{\Theta}_1\bar{\mu}_i. \end{cases} \quad (4)$$

The proof of the first inequality of (4) can be found in Proposition 3 of [7]. As for the second inequality, from Assumption 3 and Lemma 5, we know that there always exists a positive smooth function $\tilde{\varphi}_i(\bar{x}_i,\theta)$, such that $\varphi_i(\bar{x}_i,\theta) \leq \tilde{\varphi}_i(\bar{x}_i,\theta)\sum_{j=1}^{i}|x_j|$. According to Lemma 3 and Proposition B.3 in [1], there exist an unknown constant $\bar{\Theta}_1 \geq 1$ and a nonnegative smooth function $\bar{\varphi}_i(\bar{x}_i)$ satisfying $\varphi_i(\bar{x}_i,\theta) \leq \bar{\Theta}_1\bar{\varphi}_i(\bar{x}_i)\sum_{j=1}^{i}|z_j|^{\frac{1}{p_1\cdots p_{i-1}}}$. Finally, Lemma 3 guarantees the existence of nonnegative smooth functions $\bar{\mu}_i(\bar{x}_{i+1})$ and $\bar{\Theta}_1 \geq 1$ such that the last inequality holds.

Furthermore, by (4), there exist positive smooth functions $\rho_i(\bar{x}_k,\hat{\Theta})$, $l_i(\bar{x}_{k-1},\hat{\Theta})$, and an unknown constant $\bar{\Theta}_2 \geq 1$ such that for each $k = 2,\ldots,n$, $i = 1,\ldots,k-1$,

$$\begin{cases} \Big|\frac{\partial\alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial x_i}(d_i x_{i+1}^{p_i} + f_i)\Big| \leq \bar{\Theta}_2\rho_i\sum_{j=1}^{k}|z_j|, \\ \Big|\frac{\partial\alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial x_i}\Big| \cdot \|g_i\| \leq \bar{\Theta}_2 l_i \sum_{j=1}^{k-1}|z_j|. \end{cases} \quad (5)$$

Due to the limited by the space, the proof of (5) is omitted here. It should be emphasized that $\hat{\Theta}(t)$ represents appropriate estimate of an unknown positive parameter Θ defined by $\Theta \triangleq \max_{k=2,\ldots,n}\big\{\bar{\Theta}_1^2, \bar{\Theta}_2^2, \frac{1}{a}, a^{\frac{1}{2p_1\cdots p_{k-1}-1}-1}\bar{\Theta}_1^{\frac{2p_1\cdots p_{k-1}}{2p_1\cdots p_{k-1}-1}}, \frac{(a\bar{\Theta}_1)^{2p_1\cdots p_{k-2}}}{a^2}, \frac{1}{a^2}, \frac{\bar{\Theta}_1^2}{a}, \frac{\bar{\Theta}_2^2}{a}\big\}$, and $a \triangleq \min\{a_1, a_2,\ldots,a_n\}$. On the other hand, it can be deduced from (3) that

$$u(t) = -\Big(\sum_{i=1}^{n}\big(\prod_{j=i}^{n}\beta_j\big)x_i^{p_1\cdots p_{i-1}}\Big)^{\frac{1}{p_1\cdots p_n}}. \quad (6)$$

Apparently, one has to determine $\beta_1,\ldots,\beta_n$ in (6) for achieving the implementable controller $u(t)$. In the following, suppose for each $i = 1,\ldots,n$, β_i is a function of $x_1,\ldots,x_i$ and $\hat{\Theta}$, so is α_i. Then, for $k = 1,\ldots,n$, the definition of $W_k(\bar{x}_k,\hat{\Theta}) : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ is given by

$$W_k(\cdot) = \int_{\alpha_{k-1}}^{x_k}\big(s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}}\big)^{2-\frac{1}{p_1\cdots p_{k-1}}}\mathrm{d}s. \quad (7)$$

It should be noticed that (7) is used in [1] to construct Lyapunov function for the first time, which opens the door to construct continuous controller in stabilizing high-order nonlinear systems. Soon after, (7) and its various general styles are widely used in state/output feedback stabilization of high-order nonlinear systems, see [4, 6, 8, 13, 14] and references therein.

For each $k = 1,\ldots,n$, as in [4], it is not hard to prove that W_k is continuously differentiable and satisfies

$$\frac{\partial W_k}{\partial x_k} = z_k^{2-\frac{1}{p_1\cdots p_{k-1}}},\ \frac{\partial W_k}{\partial\chi_i} = -o_k F(s,\bar{\chi}_k)\frac{\partial\alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial\chi_i}, \quad (8)$$

where $F = \int_{\alpha_{k-1}}^{x_k}(s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}})^{1-\frac{1}{p_1\cdots p_{k-1}}}\mathrm{d}s$, $o_k = 2 - \frac{1}{p_1\cdots p_{k-1}}$, $\chi_i = x_i$ for $i = 1,\ldots,k-1$, and $\chi_k = \hat{\Theta}$.

Step 1. This step will specify the smooth function β_1. Consider $V_1(x_1,\tilde{\Theta}) = W_1(x_1) + \frac{a}{2}\tilde{\Theta}^2$ with $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$. According to Lemma 5, there exists a positive smooth function $\rho_0(x_1)$ that satisfies $h(x_1) \leq \rho_0(x_1)|x_1|$. It follows from (3), (4) and (8) that

$$\begin{aligned} \dot{V}_1 \ \leq\ & d_1z_1z_2 + d_1z_1\alpha_1^{p_1} + \frac{d_1}{2}|z_1x_2^{p_1}| + \bar{\Theta}_1\bar{\gamma}_1z_1^2 \\ & +\bar{\Theta}_1\bar{\varphi}_1\|\omega\|z_1^2 - a\tilde{\Theta}\dot{\hat{\Theta}} + x_1^{2p_1}\rho_0^{2p_1} - y^{2p_1}. \quad (9) \end{aligned}$$

Clearly, the application of the fact $d_1z_1\alpha_1^{p_1} \leq 0$ implies $d_1z_1\alpha_1^{p_1} + \frac{d_1}{2}|z_1x_2^{p_1}| \leq \frac{d_1}{2}|z_1z_2| + \frac{d_1}{2}z_1\alpha_1^{p_1}$, and Lemma 2 shows the inequality $\bar{\Theta}_1\bar{\varphi}_1\|\omega\|z_1^2 \leq \frac{\bar{\Theta}_1^2\bar{\varphi}_1^2z_1^4}{4\sigma} + \sigma\|\omega\|^2$, where $\sigma > 0$ is a positive design parameter. If one defines $h_1 = n + \bar{\gamma}_1 + \frac{\bar{\varphi}_1^2(1+z_1^2)}{4\sigma} + (1+z_1^2)^{p_1-1}\rho_0^{2p_1}$, $\tau_1 = h_1z_1^2$, and $\beta_1 = \frac{2h_1}{\lambda_1}\sqrt{1+\hat{\Theta}^2}$, then (9) can be rewritten as

$$\dot{V}_1 \leq -\frac{n}{a}z_1^2 + \frac{3}{2}d_1|z_1z_2| + a\tilde{\Theta}(\tau_1 - \dot{\hat{\Theta}}) - y^{2p_1} + \sigma\|\omega\|^2. \quad (10)$$

Step k $(k = 2,\ldots,n)$. Suppose at Step $k-1$, one finds a continuously differentiable Lyapunov function $V_{k-1} : \mathbb{R}^{k-1} \times \mathbb{R} \to \mathbb{R}$, and a set of positive smooth functions $\beta_1,\ldots\beta_{k-1}$ defined by (3), such that

$$\begin{aligned} \dot{V}_{k-1} \ \leq\ & -\frac{(n-k+2)}{a}\sum_{i=1}^{k-1}z_i^2 + a\tilde{\Theta}(\tau_{k-1} - \dot{\hat{\Theta}}) - y^{2p_1} \\ & +\bar{c}_{k-1}d_{k-1}|z_{k-1}|^{2-\frac{1}{p_1\cdots p_{k-2}}}|z_k|^{\frac{1}{p_1\cdots p_{k-2}}} \\ & -\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}(\tau_{k-1} - \dot{\hat{\Theta}}) + (k-1)\sigma\|\omega\|^2, \quad (11) \end{aligned}$$

where $\tau_{k-1} = \sum_{i=1}^{k-1}h_i(\bar{x}_i,\hat{\Theta})z_i^2$, h_i is a well-defined smooth function, and $\bar{c}_{k-1}$ is a known positive constant. It is apparent that (11) reduces to (10) when $k = 2$. In this step, what needs to be done is claiming the existence of β_k. Taking the time derivative of $V_k = V_{k-1} + W_k$ along solutions of (1), the use of (8) and (11) yields

$$\begin{aligned} \dot{V}_k \leq & -\frac{(n-k+2)}{a}\sum_{i=1}^{k-1}z_i^2 + a\tilde{\Theta}(\tau_{k-1} - \dot{\hat{\Theta}}) + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\dot{x}_i \\ & +\frac{\partial W_k}{\partial\hat{\Theta}}\dot{\hat{\Theta}} + z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}(d_kx_{k+1}^{p_k} + f_k + g_k\omega) - y^{2p_1} \\ & -\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}(\tau_{k-1} - \dot{\hat{\Theta}}) - \sigma\|\omega\|^2 + k\sigma\|\omega\|^2 \end{aligned}$$

$$+\bar{c}_{k-1}d_{k-1}|z_{k-1}|^{2-\frac{1}{p_1\cdots p_{k-2}}}|z_k|^{\frac{1}{p_1\cdots p_{k-2}}}. \tag{12}$$

First, it follows from (4) and Lemma 2 that

$$\begin{aligned}&\bar{c}_{k-1}d_{k-1}|z_{k-1}|^{2-\frac{1}{p_1\cdots p_{k-2}}}|z_k|^{\frac{1}{p_1\cdots p_{k-2}}}\\ &\leq \bar{c}_{k-1}\bar{\Theta}_1\bar{\mu}_{k-1}|z_{k-1}|^{2-\frac{1}{p_1\cdots p_{k-2}}}|z_k|^{\frac{1}{p_1\cdots p_{k-2}}}\\ &\leq a\Theta h_{k1}(\bar{x}_k)z_k^2+\frac{1}{4a}\sum_{i=1}^{k-1}z_i^2,\end{aligned} \tag{13}$$

where $h_{k1}=\frac{(4p_1\cdots p_{k-2}-2)^{2p_1\cdots p_{k-2}-1}}{2(p_1\cdots p_{k-2})^{2p_1\cdots p_{k-2}}}(\bar{c}_{k-1}\bar{\mu}_{k-1})^{2p_1\cdots p_{k-2}}$. Second, positive constant $\bar{c}_k$ and positive smooth function $h_{k2}(\bar{x}_k,\hat{\Theta})$ guarantee

$$\begin{aligned}&z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}(d_kx_{k+1}^{p_k}+f_k+g_k\omega)\\ &\leq \frac{1}{2}d_kz_k^{2-\frac{1}{p_1\cdots p_{k-1}}}\alpha_k^{p_k}+a\Theta h_{k2}z_k^2+\frac{1}{4a}\sum_{i=1}^{k-1}z_i^2-\frac{n}{a}z_k^2\\ &+\bar{c}_kd_k|z_k|^{2-\frac{1}{p_1\cdots p_{k-1}}}|z_{k+1}|^{\frac{1}{p_1\cdots p_{k-1}}}+\frac{\sigma}{2}\|\omega\|^2.\end{aligned} \tag{14}$$

The proof of (14) is also omitted here. Third, for $i=1,\ldots,k$, it can be easily verified that

$$\left|\frac{\partial W_k}{\partial \chi_i}\right|\leq c_k|z_k|\cdot\left|\frac{\partial \alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial \chi_i}\right|, \tag{15}$$

where $c_k=\left(2-\frac{1}{p_1\cdots p_{k-1}}\right)2^{1-\frac{1}{p_1\cdots p_{k-1}}}$, so it can be seen from (5), (15), Lemmas 1 and 2 that

$$\begin{aligned}&\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\dot{x}_i=\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(d_ix_{i+1}^{p_i}+f_i+g_i\omega)\\ &\leq c_k|z_k|\sum_{i=1}^{k-1}\left|\frac{\partial \alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial x_i}\right|\cdot\left(|d_ix_{i+1}^{p_i}+f_i|+\|g_i\|\cdot\|\omega\|\right)\\ &\leq a\Theta h_{k3}(\bar{x}_k,\hat{\Theta})z_k^2+\frac{1}{4a}\sum_{i=1}^{k-1}z_i^2+\frac{\sigma}{2}\|\omega\|^2,\end{aligned} \tag{16}$$

where $h_{k3}=c_k\sum_{i=1}^{k-1}\rho_i+(k-1)c_k^2(\sum_{i=1}^{k-1}\rho_i)^2+\frac{1}{2\sigma}2^{k-1}c_k^2(\sum_{i=1}^{k-1}l_i)^2\sum_{i=1}^{k-1}z_i^2$. Let $h_k=h_{k1}+h_{k2}+h_{k3}$, $\tau_k=\tau_{k-1}+h_kz_k^2$. Furthermore, Lemma 2 shows

$$\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial \hat{\Theta}}h_kz_k^2+\frac{\partial W_k}{\partial \hat{\Theta}}\tau_k\leq a\bar{h}_kz_k^2+\frac{k-1}{a}z_k^2+\frac{1}{4a}\sum_{i=1}^{k-1}z_i^2,$$

where $\bar{h}_k(\bar{x}_k,\hat{\Theta})=\frac{1}{2(k-1)}h_k^2\left(k-2+\sum_{i=2}^{k-1}(\frac{\partial W_i}{\partial \hat{\Theta}})^2\right)^2+\left(1+(\frac{\partial \alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial \hat{\Theta}})^2\right)\left(\frac{1}{2(k-1)}h_k^2z_k^2+c_k^2\sum_{i=1}^{k-1}h_i^2z_i^2\right)$. Now, if one chooses the smooth function $\beta_k=\left(\frac{2}{\lambda_k}(h_k\sqrt{1+\hat{\Theta}^2}+\bar{h}_k)\right)^{p_1\cdots p_{k-1}}$, and substituting (13), (14) and (16) into (12), then

$$\begin{aligned}\dot{V}_k\leq&-\frac{(n-k+1)}{a}\sum_{i=1}^{k}z_i^2+a\tilde{\Theta}(\tau_k-\dot{\hat{\Theta}})-y^{2p_1}+k\sigma\|\omega\|^2\\ &+\bar{c}_kd_k|z_k|^{2-\frac{1}{p_1\cdots p_{k-1}}}|z_{k+1}|^{\frac{1}{p_1\cdots p_{k-1}}}-\sum_{i=2}^{k}\frac{\partial W_i}{\partial \hat{\Theta}}(\tau_k-\dot{\hat{\Theta}}),\end{aligned}$$

which still holds for $k=n$ with $z_{n+1}=0$. In other words, once $\beta_1,\ldots,\beta_n$ are assigned appropriately, the actual adaptive controller is constructed as follows.

$$\begin{cases}u(t)=-\left(\sum_{i=1}^{n}\left(\prod_{j=i}^{n}\beta_j\right)x_i^{p_1\cdots p_{i-1}}\right)^{\frac{1}{p_1\cdots p_n}},\\ \dot{\hat{\Theta}}=\sum_{i=1}^{n}h_i\left(x_i^{p_1\cdots p_{i-1}}+\sum_{j=1}^{i-1}(\prod_{l=j}^{i-1}\beta_l)x_j^{p_1\cdots p_{j-1}}\right)^2.\end{cases} \tag{17}$$

With the definition of $\sigma=\frac{\varepsilon^2}{n}$, there is

$$\dot{V}_n+y^{2p_1}-\varepsilon^2\|\omega\|^2\leq-\frac{1}{a}\sum_{i=1}^{n}z_i^2, \tag{18}$$

where $V_n(x,\tilde{\Theta})=\sum_{i=1}^{n}W_i+\frac{a}{2}\tilde{\Theta}^2$.

Part II: theoretical analysis

It can be observed that the closed-loop systems are composed of (1) and (17). Theorem 2.1 in [22] says that the closed-loop system states can be defined on a time interval $[0,\,t_m)$ for some $t_m>0$, since $h_i,\beta_i,f_i,g_i,d_i,i=1,\ldots,n$ and ω are continuous nonlinear functions. We are interested in $t_m=\infty$. As a matter of fact, there are exist positive constants c_{k1} and c_{k2} such that

$$c_{k1}|x_k-\alpha_{k-1}|^{2p_1\cdots p_{k-1}}\leq W_k\leq c_{k2}z_k^2,\ k=1,\ldots,n. \tag{19}$$

In light of it, one can prove that V_n is positive definite and $V_n(x,\tilde{\Theta})\to\infty$ as $\|[x,\tilde{\Theta}]^T\|\to\infty$. Moreover, (18) implies

$$\dot{V}_n(x(t),\tilde{\Theta}(t))\quad\leq\quad\varepsilon^2\|\omega(t)\|^2,\ \forall t\in[0,t_m). \tag{20}$$

Integrating above inequality from 0 to t, it can be deduced from $\omega\in L_2$ that

$$\begin{aligned}&V_n(x(t),\tilde{\Theta}(t))\leq V_n(x(0),\tilde{\Theta}(0))+\varepsilon^2\int_0^t\|\omega(s)\|^2\mathrm{d}s\\ &\leq V_n(x(0),\tilde{\Theta}(0))+\varepsilon^2\int_0^\infty\|\omega(s)\|^2\mathrm{d}s<\infty.\end{aligned} \tag{21}$$

The upper bound on $V_n(x(t),\tilde{\Theta}(t))$ is finite for every finite t, and it approaches infinity probably only as $t\to\infty$. The states of the closed-loop systems cannot have a finite escape time, for if it were not true, there would be a finite time $t_1\in[0,t_m)\bigcap[0,\infty)$, such that $\lim_{t\to t_1}\|[x(t),\tilde{\Theta}(t)]^T\|\to\infty$, so $V_n(x(t),\tilde{\Theta}(t))\to\infty$ as $t\to t_1$, this is a contradiction. Therefore, $t_m=\infty$, in other words, the states are defined for all $t\geq0$. The left proof is divided into two parts.

(i) When $\omega=0$. (21) implies $W_i\in L_\infty$ and $\tilde{\Theta}\in L_\infty$. With $\hat{\Theta}=\Theta-\tilde{\Theta}$, one has $\hat{\Theta}\in L_\infty$. Noticing $W_1=\frac{x_1^2}{2}$, so $x_1\in L_\infty$, which also shows $\alpha_1=-(\beta_1(x_1,\hat{\Theta})x_1)^{\frac{1}{p_1}}\in L_\infty$, since continuous function β_1 is bounded for $x_1\in L_\infty$ and $\hat{\Theta}\in L_\infty$. Assisted by (19) and Lemma 1, there is $x_2\leq|x_2-\alpha_1|+|\alpha_1|\leq\left(W_2/c_{k1}\right)^{\frac{1}{2p_1}}+|\alpha_1|\in L_\infty$. In this way, one can prove $x_3\in L_\infty,\ldots,x_n\in L_\infty$, and it follows from (17) that $u\in L_\infty$. So far, the states $[x(t),\hat{\Theta}(t)]^T$ of the closed-loop systems and control input $u(t)$ are globally uniformly bounded on the interval $[0,\infty)$. To prove $\lim_{t\to\infty}x(t)=0$, we easily see from

(20) that $\dot{V}_n(x(t), \tilde{\Theta}(t)) \leq 0$, $V_n(x(t), \tilde{\Theta}(t))$ is monotonically nonincreasing and bounded lower by zero, and hence $\lim_{t\to\infty} V_n(x(t), \tilde{\Theta}(t))$ exists and is finite. Thus,

$$0 \leq \int_0^{\infty} z_i^2(s)\mathrm{d}s \leq aV_n(x(0), \tilde{\Theta}(0)) < \infty, \quad (22)$$

which means $z_i \in L_2$, $i = 1, \dots, n$. Clearly, there is $x_1 \in L_2 \bigcap L_\infty$, and $\dot{x}_1 \in L_\infty$ can be achieved by $x \in L_\infty$ and $u \in L_\infty$, hence, Lemma 4 shows $\lim_{t\to\infty} x_1(t) = 0$. Furthermore, to prove $\lim_{t\to\infty} x_2(t) = 0$, since $\beta_1 \in L_\infty$ promises $\beta_1^2(x_1, \hat{\Theta}) \leq \Xi$ with a positive constant Ξ, one immediately deduces from Lemma 1 that $\int_0^\infty x_2^{2p_1}(s)\mathrm{d}s \leq 2\int_0^\infty z_2^2(s)\mathrm{d}s + 2\Xi^2\int_0^\infty z_1^2(s)\mathrm{d}s < \infty$, hence $x_2 \in L_{2p_1}$. Then, Lemma 4 shows $\lim_{t\to\infty} x_2(t) = 0$ again. Similarly, there are $\lim_{t\to\infty} x_i(t) = 0, i = 3, \dots, n$, which in turn lead to $\lim_{t\to\infty} x(t) = 0$.

(ii) When $\omega \neq 0$. Performing the procedures in the case of $\omega = 0$, one can still claim that the states of the closed-loop systems are globally uniformly bounded on $[0, \infty)$. However, the convergence of $x(t)$ is replaced by the following estimate. It can be deduced from (18) that

$$\int_0^t |y(s)|^{2p_1}\mathrm{d}s \leq \varepsilon^2\int_0^t \|\omega(s)\|^2\mathrm{d}s - \int_0^t \Big(\dot{V}_n(x(s), \tilde{\Theta}(s)) + \frac{1}{a}\sum_{i=1}^{n} z_i^2(s)\Big)\mathrm{d}s \leq \varepsilon^2\int_0^t \|\omega(s)\|^2\mathrm{d}s + \delta(x(0), \hat{\Theta}(0)),$$

where $\delta(x(0), \hat{\Theta}(0)) = V_n(x(0), \tilde{\Theta}(0))$, and t can be ∞. This completes the proof. □

Theorem 2. Consider the high-order uncertain nonlinear systems (1) under Assumptions 1-3. If $\omega \in L_{2m}$ with m being a positive integer, the ADA problem is solvable.

Proof. The proof of Theorem 2 is reminiscent of the proof of Theorem 1, we only modify some inequalities without changing the construction of V_i. For instance, to estimate the term $\bar{\Theta}_1\bar{\varphi}_1\|\omega\|z_1^2$ in (9) in Step 1, the inequality is adjusted to $\bar{\Theta}_1\bar{\varphi}_1\|\omega\|z_1^2 \leq \frac{2m-1}{2m}\big(\frac{1}{2m\eta}\big)^{\frac{1}{2m-1}}(\bar{\Theta}_1\bar{\varphi}_1 z_1^2)^{\frac{2m}{2m-1}} + \eta\|\omega\|^{2m}$. Meanwhile, h_1 and Θ are respectively amended to $h_1 = \frac{2m-1}{2m}(\frac{1}{2m\eta})^{\frac{1}{2m-1}}\bar{\varphi}_1^{\frac{2m}{2m-1}} z_1^{\frac{2}{2m-1}} + \bar{\gamma}_1 + (1 + z_1^2)^{2mp_1-2}\rho_0^{2mp_1} + n$, $\Theta \triangleq \max\{\frac{1}{a}, \frac{1}{a^2}, \frac{\bar{\Theta}_1}{a}, \frac{1}{a}\bar{\Theta}_1^{\frac{2m}{2m-1}}\}$. □

Remark 2. Theorems 1 and 2 unify control design in the existing papers. It is worthwhile emphasizing that they reduce to the counterparts in [4, 7, 20, 21] by letting $\theta = 0$, $\omega = 0$ and $p_i = 1$, respectively. In addition, because of the appearance of unknown parameters and disturbance signals, there will be much more nonlinear terms than those in [4, 7, 20, 21], how to deal with them and introduce a reasonable unknown parameter constitutes one of the main contributions of this paper.

4 Simulation Example

Consider the following uncertain nonlinear system

$$\begin{cases} \dot{x}_1 = (2 - 0.2\sin(\theta t))x_2^{\frac{5}{3}} + \theta x_1\cos t + \theta x_1\omega, \\ \dot{x}_2 = u - 2\theta\sin x_2 + \theta\omega\sin x_1, \end{cases}$$

where system output is $y = x_1$, θ is an unknown parameter, and $\omega \in R$ is a disturbance signal. It is easy to verify that the system satisfies Assumptions 1-3 with $\lambda_1 = 1.8$, $\mu_1 = 2.2$, $\lambda_2 = 1$, $\mu_2 = 1$, $\varphi_1 = \varphi_2 = |\theta x_1|$, $a_1 = a_2 = 1$. By the definition of $\Theta \triangleq \max\{\bar{\Theta}_1^2, \bar{\Theta}_2^2, \frac{1}{a}, \frac{1}{a^2}, \frac{\bar{\Theta}_1}{a}, \frac{\bar{\Theta}_1^2}{a}, a^{\frac{1}{2p_1-1}-1}\bar{\Theta}_1^{\frac{2p_1}{2p_1-1}}\} = \max\{\bar{\Theta}_1^2, \bar{\Theta}_2^2\}$ due to $a = \min\{a_1, a_2\} = 1$, we obtain the actual adaptive controller (17) with $n = 2$, where $\beta_1 = \frac{2h_1}{\lambda_1}\sqrt{1 + \hat{\Theta}^2}$, $\beta_2 = \big(\frac{2}{\lambda_2}(h_2\sqrt{1 + \hat{\Theta}^2} + \bar{h}_2)\big)^{\frac{5}{3}}$, $h_1 = \bar{\gamma}_1 + \frac{\bar{\varphi}_1^2(1+x_1^2)}{4\sigma} + (1 + x_1^2)^{\frac{2}{3}}\rho_0^{\frac{10}{3}} + 2$, $h_2 = \frac{9}{4}\bar{\mu}_1^2 + \bar{\gamma}_2 + 2 + \frac{7}{10}(\frac{6}{5})^{\frac{3}{7}}\bar{\gamma}_2^{\frac{10}{7}} + \frac{3}{\sigma}\bar{\varphi}_2^2(x_2^{\frac{5}{3}} + \beta_1 x_1)^2 + c_2\rho_1 + c_2^2\rho_1^2 + \frac{1}{\sigma}c_2^2 l_1^2 x_1^2 + \frac{2}{\beta p_1}\bar{\varphi}_2^2(x_1^2 - (x_2^{\frac{5}{3}} + \beta_1 x_1)^2)$, and $\bar{h}_2 = c_2^2\big(1 + (\frac{\partial\alpha_1^{p_1}}{\partial\hat{\Theta}})^2\big)\big(h_1^2 x_1^2 + \frac{1}{4}h_2^2(x_2^{\frac{5}{3}} + \beta_1 x_1)^2\big)$, with $c_2 = \frac{7}{5}\cdot 2^{\frac{2}{5}}$, $\bar{\gamma}_1 = \bar{\varphi}_1 = \bar{\varphi}_2 = \rho_0 = 1$, $\bar{\gamma}_2 = 4$, $l_1 = \beta_1$, $\rho_1 = 2\beta_1$, $\frac{\partial\alpha_1^{p_1}}{\partial\hat{\Theta}} = -\frac{2h_1\hat{\Theta}}{\sqrt{1+\hat{\Theta}^2}\lambda_1}$. In simulation, we select $\theta = 1$, and set the initial conditions as $x_1(0) = 1, x_2(0) = -2, \hat{\Theta}(0) = 1$. Figs 1–4 illustrate the effectiveness of the controller.

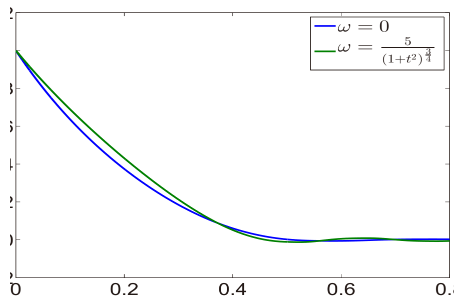

Fig. 1 The trajectories of the state x_1.

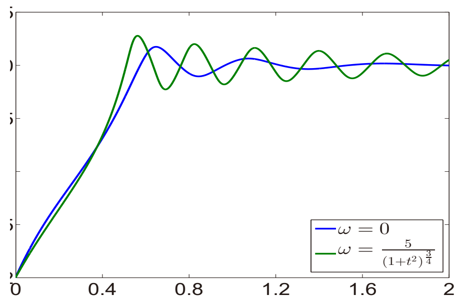

Fig. 2 The trajectories of the state x_2.

5 Conclusions

In this paper, we find the feasible conditions and provide a constructive solution to AADD for generalized high-order uncertain nonlinear systems. The designed adaptive controller guarantees global stabilization properties in the absence of disturbance, and attenuates the influence of the disturbance on the output with an arbitrary degree of accuracy in terms of L_2-L_{2p} gain. The difficulty lies in managing coupling between unknown parameters and disturbance signals.

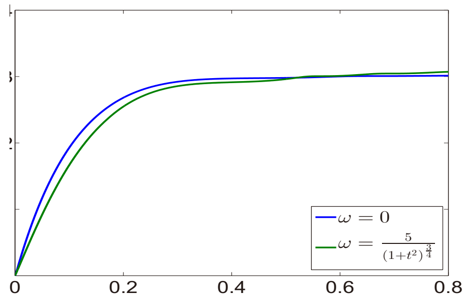

Fig. 3 The trajectories of $\hat{\Theta}$.

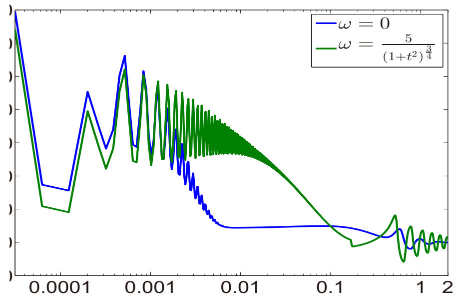

Fig. 4 The trajectories of controller u.

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