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Lyapunov-based adaptive model predictive control for unconstrained non-linear systems with parametric uncertainties

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Abstract: In this study, a simple Lyapunov-based adaptive model predictive control (MPC) is proposed to stabilise a class of unconstrained non-linear systems with constant parametric uncertainties. In the proposed MPC design, the uncertain parameters are estimated online with an adaptive updating law, and the estimated parameters are guaranteed bounded. A Lyapunov-based constraint is employed in the adaptive MPC to ensure the stability of the closed-loop system. By using the control Lyapunov function-based constraint, terminal penalties in traditional MPC can be avoided, such that computational burden is significantly reduced. Both theoretical results and numerical examples demonstrate that, with the proposed adaptive MPC, states of the closed-loop system can be stabilised, while the adaptive estimated parameters are bounded.

1 Introduction

Model predictive control (MPC) has been prevailingly used to address process control problems (with or without constraints). For an overview on MPC, please refer to the survey paper [1]. The main reasons that MPC can be widely applied include: (i) superior robustness with respect to external disturbances and (ii) explicit constraint handling [2]. In MPC design, the system outputs at the next several sampling times are predicted by using the system state equation, and are fed-back for calculating controls for the corresponding sampling times. The actual control is implemented by a receding horizon way. In this way, the long term effect of disturbances is considered in control design, contributing to the superior robustness of MPC. The explicit constraint handling can be performed by using numerical optimisation techniques. To deal with specific industrial projects, some theoretical and technological variations of classical MPC have been developed in recent years. For example, MPC with disturbance feedback [3], MPC for switched non-linear systems [4], time-varying MPC [5], robust MPC [6], and networked predictive control [7]. Applications of MPC to various areas include energy generation [8], resource allocation [9], chemical process [10], flight control [11] and so on.

Although its inherent robustness with respect to external disturbances is usually satisfactory, the performance of MPC with respect to parametric uncertainties still remains an open topic (at least theoretically). This is because parametric uncertainties would lead to difficulties in predicting future states of the plant. An intuitive solution to the problem of MPC design with parametric uncertainties is to introduce adaptive strategies, such that the prediction can still be processed with estimated parameters instead of uncertain parameters. Early researches on adaptive MPC can date back to [12], where implicit updating laws are proposed to estimate the constant parametric uncertainties. Some other representative researches include adaptive MPC based on persistent excitation [13], adaptive strategy for single-loop MPC [14], adaptive MPC by using comparison model [15], adaptive MPC for constrained discrete-time linear systems [16], and adaptive MPC for constrained continuous non-linear systems [17]. Recently, neural networks are introduced in adaptive MPC design to solve problems of system identification [18] and time delay [19].

In this paper, a new simple adaptive MPC is proposed for a class of unconstrained non-linear systems with constant parametric uncertainties. The proposed adaptive MPC is developed by combining an adaptive updating law and a constrained MPC based on control Lyapunov function. The theoretical framework of the proposed adaptive MPC is enlightened by Lyapunov-based MPC [4, 20], where control Lyapunov function is constructed and regarded as an extra non-linear constraint. The primary advantage of Lyapunov-based MPC is that, using a Lyapunov function as an extra non-linear constraint, stability of the closed-loop system can be guaranteed by the feasibility of the optimisation problem. The main *contributions* of this paper include that, (i) by using a Lyapunov-based non-linear constraint, a simple formulation of MPC can be proposed such that prediction can be performed in case of parametric uncertainties, and stability of the closed-loop system can be guaranteed; and (ii) no terminal constraints are necessary in the proposed adaptive MPC to ensure stability, such that the computational burden in optimisation is significantly reduced.

The configuration of this paper is arranged as following: the problem of designing adaptive MPC is stated in Section 2; main results of the adaptive MPC for non-linear system with constant parametric uncertainties is proposed in Section 3; an extension of the proposed adaptive MPC to include soft constraints is discussed in Section 4; two numerical simulation examples of the closed-loop system with the proposed adaptive MPC are presented in Section 5; concluding remarks are given in the final section.

2 Problem statement

In this paper, we consider the non-linear single-input system as

$$\dot{x} = f(x) + g(x)u + \Phi(x)\theta, \tag{1}$$

where $x \in \mathcal{R}^n$ and $u \in \mathcal{R}$ are system states and control input, respectively; $f(x) \in \mathcal{R}^n$ and $g(x) \in \mathcal{R}^n$ are known continuously differentiable functions with respect to x satisfying $f(0) = 0$ and $g(x) \neq 0$; $\Phi(x) = [\phi_1(x), \phi_2(x), \dots, \phi_k(x)]$ is a known continuously differentiable vector function satisfying $\Phi(0) = 0$; and $\theta = [\theta_1, \theta_2, \dots, \theta_k]^{\mathrm{T}}$ denotes the vector of uncertain constant parameters.

2.1 Assumptions

Assumption 1 (full-state feedback linearisable [21]): For system (1), the matrix

$$\mathcal{G} = \left[g, \mathrm{ad}_f g, \mathrm{ad}_f^2 g, \ldots, \mathrm{ad}_f^{n-1} g\right] \tag{2}$$

is of rank n, and the distribution

$$\mathcal{D} = \mathrm{span}\left\{g, \mathrm{ad}_f g, \mathrm{ad}_f^2 g, \ldots, \mathrm{ad}_f^{n-2} g\right\}$$

is involutive [21], where

$$\mathrm{ad}_f^i g \triangleq \left[f, \mathrm{ad}_f^{i-1} g\right], \ \mathrm{ad}_f g \triangleq [f, g], \ [f, g] \triangleq \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g. \tag{3}$$

It is indicated from Assumption 1 that, there exist a global diffeomorphism $z = [z_1, \ldots, z_n]^{\mathrm{T}} = H(x)$ satisfying $H(0) = 0$, and a state feedback control $u = \alpha(x) + \beta(x)v$, such that the nominal system

$$\dot{x} = f(x) + g(x)u \tag{4}$$

can be transferred into

$$\begin{aligned} &\dot{z}_i = z_{i+1}, \ 1 \le i \le n-1; \\ &\dot{z}_n = v. \end{aligned} \tag{5}$$

It can be seen that (5) is in a chained integral form, and there exist a continuously differentiable feedback control $v = \kappa(z)$ to stabilise it exponentially. It then follows from converse Lyapunov theorem that a Lyapunov function $V(z)$ exists, such that

$$\alpha_1 \parallel z \parallel^2 \leqslant V(z) \leqslant \alpha_2 \parallel z \parallel^2, \tag{6}$$

$$\dot{V}(z) \leqslant -\alpha_3 \parallel z \parallel^2, \tag{7}$$

$$\parallel \frac{\partial V(z)}{\partial z} \parallel \leqslant \alpha_4 \parallel z \parallel, \tag{8}$$

where α_1, α_2, α_3, α_4 are positive constants; and $\parallel \cdot \parallel$ denotes the Euclidean norm for vectors. We can rewrite (7) into a more explanatory form:

$$\begin{aligned} \dot{V}(z) &= \frac{\partial V(z)}{\partial z}\dot{z} = \frac{\partial V(z)}{\partial z}\frac{\partial z}{\partial x}(f(x) + g(x)\kappa(H(x))) \\ &\leqslant -\alpha_3 \parallel z \parallel^2. \end{aligned} \tag{9}$$

It should be noted that, in (7), α_3 can be tuned arbitrarily large by selecting appropriate $\kappa(z)$.

Assumption 2: Although the exact value of θ is unknown, a conservative estimation of its bound can be known

$$\parallel \theta \parallel \le \bar{\theta}, \tag{10}$$

where $\bar{\theta} > 0$ is a positive real number.

2.2 Control objective

Define a cost function for the nominal system (4)

$$J_0 = \int_t^{t+T} l(x(\tau|t), u(\tau|t))\mathrm{d}\tau, \tag{11}$$

where T denotes the prediction horizon; $l(x, u)$ is a positive definite scalar function with respect to x and u, and $l(0, 0)$ reaches its minimum value; $x(\tau|t)$ and $u(\tau|t)$ are the predictive states and input at time t. In this paper, the control horizon is assigned the same as predictive horizon for simplicity.

The traditional MPC for the nominal system (4) can be designed by

$$u(t) = u^*(\tau|t), \ \tau = t, \tag{12}$$

where $u^*(\tau|t)$ is the solution of the optimisation problem $\mathcal{P}_{\mathrm{nom}}$:

$$\min_{u(\cdot|t)} J_0 \tag{13}$$

subject to the dynamic constraint

$$\dot{x}(\tau|t) = f(x(\tau|t)) + g(x(\tau|t))u(\tau|t), \ \tau \in [t, t+T). \tag{14}$$

In this paper, hard constraints on system states and input in forms of $x \in \mathcal{X}$ and $u \in \mathcal{U}$ are beyond consideration.

For the non-linear system (1) with constant uncertain parameters, its dynamic equation cannot be directly used as dynamic constraint (predictive equation). An intuitive solution is to find an adaptive strategy to estimate the uncertain parameters, such that prediction can be proceeded.

To this end, the *objective* of this paper is to design an adaptive MPC for system (1) with constant uncertain parameters, such that

$$\lim_{t \to \infty} \parallel x(t) \parallel < \epsilon, \tag{15}$$

where $\epsilon > 0$ is a small positive real number.

3 Main results

The adaptive MPC consists two parts: an adaptive estimator and an MPC for the estimated system.

3.1 Adaptive updating law for the estimated parameter

Define an estimated parameter $\hat{\theta}$ for the uncertain constant parameter, and define the estimated error $\tilde{\theta} \triangleq \theta - \hat{\theta}$. Select a Lyapunov function

$$\hat{V} = V(z) + \frac{1}{2\gamma}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} = V(H(x)) + \frac{1}{2\gamma}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}, \tag{16}$$

where $V(z)$ satisfies (6)–(8); and γ is a positive real number. For simplicity, $V(H(x))$ will be written as $V(x)$ in the following sections.

Remark 1: The Lyapunov function $V(H(x))$ is selected based on the nominal system (4), and it is independent of the uncertain parameter θ.

The derivative of the Lyapunov candidate can be calculated by

$$\begin{aligned} \dot{\hat{V}} &= \frac{\partial V(x)}{\partial x}\dot{x} - \frac{1}{\gamma}\tilde{\theta}^{\mathrm{T}}\dot{\hat{\theta}} \\ &= \frac{\partial V(x)}{\partial x}(f(x) + g(x)u + \Phi(x)\theta) - \frac{1}{\gamma}\tilde{\theta}^{\mathrm{T}}\dot{\hat{\theta}} \\ &= \frac{\partial V(x)}{\partial x}\left(f(x) + g(x)u + \Phi(x)\hat{\theta}\right) + \frac{\partial V(x)}{\partial x}\Phi(x)\tilde{\theta} - \frac{1}{\gamma}\tilde{\theta}^{\mathrm{T}}\dot{\hat{\theta}}. \end{aligned} \tag{17}$$

Let the adaptive updating law be designed by

$$\dot{\hat{\theta}} = \mathrm{Prj}_{\hat{\theta}}\left(\gamma\Phi^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\right), \tag{18}$$

where $\mathrm{Prj}_{\hat{\theta}}(\cdot)$ is the projection operator [22] with respect to the bound $\bar{\theta}$, as is defined by

$$\mathrm{Prj}_{\hat{\theta}}\left(\gamma\Phi^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\right) = \begin{cases} \gamma\Phi^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}} \text{if } \|\hat{\theta}\| < \bar{\theta}, \\ \quad \text{or } \|\hat{\theta}\| = \bar{\theta}, \text{ and } \hat{\theta}^{\mathrm{T}}\Phi^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}} < 0; \\ \gamma\Phi^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}} - \gamma\frac{\hat{\theta}^{\mathrm{T}}\Phi^{\mathrm{T}}(x)(\partial V(x)/\partial x)^{\mathrm{T}}}{\|\hat{\theta}\|^2}\hat{\theta} \\ \quad \text{if } \|\hat{\theta}\| = \bar{\theta}, \text{ and } \hat{\theta}^{\mathrm{T}}\Phi^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}} \geq 0. \end{cases} \tag{19}$$

The projection operator-based adaptive law guarantees that, if the initial value of $\hat{\theta}$ satisfies that $\|\hat{\theta}(0)\| \leq \bar{\theta}$, then

$$\|\hat{\theta}(t)\| \leq \bar{\theta}, \tag{20}$$

and

$$\dot{V} \leq \frac{\partial V(x)}{\partial x}\left(f(x) + g(x)u + \Phi(x)\hat{\theta}\right), \tag{21}$$

where the control $u = \kappa(x, \hat{\theta})$ is to be designed with MPC approach to ensure that $\dot{V} \leqslant 0$. Proofs of (20) and (21) are given in the Appendix.

3.2 Lyapunov-based adaptive MPC

Design the stage cost function

$$l(x,u) = \|x\|_P^2 + \|u\|_Q^2, \tag{22}$$

where t denotes the current sampling time, T denotes the predictive horizon; P and Q are positive definite matrices such that $\|x\|_P^2 = x^{\mathrm{T}}Px$ and $\|u\|_Q^2 = u^{\mathrm{T}}Qu$. The cost function J_0 is defined by (11).

Remark 2: In this paper, the control horizon is assigned equal to the predictive horizon:

$$T_c = T_p = T. \tag{23}$$

The open-loop optimal problem at each sampling time t can be formulated as $\mathcal{P}_l$:

$$\min_{u(\cdot|t)} J_0, \tag{24}$$

subject to constraints:

$$\hat{\theta}(\tau|t) = \hat{\theta}(t),\ \tau \in [t, t+T), \tag{25}$$

$$x(\tau|t) = x(t),\ \tau = t, \tag{26}$$

$$\dot{x}(\tau|t) = f(x(\tau|t)) + g(x(\tau|t))u(\tau|t)) + \Phi(x(\tau|t))\hat{\theta}(\tau|t), \quad \tau \in [t, t+T), \tag{27}$$

$$\frac{\partial V}{\partial x}(f(x(\tau|t)) + g(x(\tau|t)u(\tau|t))) \leqslant -\alpha_3(\tau)\|H(x(\tau|t))\|^2, \quad \tau = t, \tag{28}$$

where $\hat{\theta}(t)$ is updated by (18); $V(H(x))$ satisfies (6) and (8); the time-varying $\alpha_3(t)$ can be obtained by solving the following optimisation problem $\mathcal{P}_\alpha$:

$$\min_{\alpha_3} \alpha_3^2, \tag{29}$$

subject to:

$$\alpha_3 > 0, \tag{30}$$

(see (31))

$$\alpha_3 = \delta, \text{ if } \|z(t)\| = 0, \tag{32}$$

where $\delta > 0$ can be assigned according to the requirements of transient performances.

The adaptive MPC can be proceeded by implementing

$$u(t) = u^*(\tau|t), \quad \tau = t, \tag{33}$$

where $u^*(\cdot|t)$ is the optimal solution of $\mathcal{P}_l$.

Remark 3: The constraint (25) indicates that, after updating at each sampling time, the adaptive estimated parameter remains constant during the entire control horizon.

Remark 4: The constraint (28) is required only at the initial predictive stage $\tau = t$, but not during the entire control horizon, such that computational burden could be significantly reduced. The reason is that, in MPC (or receding horizontal control), only predictive control $u(\tau|t)$ at $\tau = t$ is implemented, while the predictive controls in other steps of the control horizon ($\tau \in (t, t+T)$) are discarded.

The Lyapunov-based adaptive MPC algorithm can be summarised as following:

Algorithm 1:

1. at the sampling time t, update the estimated parameter $\hat{\theta}(t)$ with the adaptive updating law (18);
2. calculate $\alpha_3(t)$ by solving the optimisation problem (29) subject to the constraint (31);
3. solve the optimal problem (24) subject to constraints (25)–(28), and obtain the optimal solution u^*;
4. implement adaptive MPC by (33);
5. repeat steps (1)–(4).

3.3 Feasibility and stability

Feasibility of optimisation and stability of the closed-loop system with the proposed adaptive MPC can be given in the following theorem.

Theorem 1: For non-linear system (1) with parametric uncertainties satisfying Assumptions 1 and 2, it holds that

1. (feasibility) the optimisation problems $\mathcal{P}_\alpha$ and $\mathcal{P}_l$ are feasible;
2. (stability) the closed-loop system with Algorithm 1 is globally asymptotically stable.

Proof:

i. Optimisation problem $\mathcal{P}_\alpha$ is a convex optimisation if $\|z(t)\| \neq 0$; consequently, the optimisation is feasible.

   For problem $\mathcal{P}_l$, constraints (25)–(27) are for prediction; they are satisfied naturally at all time.

$$\alpha_3\|z(t)\|^2 \geq \left\|\frac{\partial V}{\partial z}\frac{\partial z}{\partial x}\Phi(x(t))\hat{\theta}(t)\right\| + \delta\|z(t)\|^2, \text{ if } \|z(t)\| \neq 0, \tag{31}$$

With Assumption 1 and the corresponding derivations given in Section 2.1, for any positive $\alpha_3(t)$, there always exists $u(x(\tau|t)) = \kappa(H(x(\tau|t)))$ at $\tau = t$, such that

$$\begin{aligned}\frac{\partial V(H(x(\tau|t)))}{\partial H(x(\tau|t))}\frac{\partial H(x(\tau|t))}{\partial x(\tau|t)}&(f(x(\tau|t))\\ &+g(x(\tau|t))\kappa(H(x(\tau|t)))\\ &\leqslant -\alpha_3(\tau)\parallel H(x(\tau|t))\parallel^2,\end{aligned} \tag{34}$$

indicating that, there always exists a control flow $u(\cdot|t)$, such that the Lyapunov-based constraint (28) can be satisfied. Moreover, for arbitrarily large α_3, (28) can be satisfied by selecting appropriate κ. Consequently, feasibility of $\mathcal{P}_l$ is proved.

ii. Since $x(t|t) = x(t)$ and $u(t|t) = u(t)$, it follows from the Lyapunov constraint (28) that

$$\frac{\partial V(x(t))}{\partial x(t)}(f(x(t)) + g(x(t))u(t)) \leqslant -\alpha_3(x)\parallel H(x(t))\parallel^2. \tag{35}$$

Substituting (35) into (21) yields that

$$\dot{\hat{V}} \leq -\alpha_3(x)\parallel H(x(t))\parallel^2 + \parallel \frac{\partial V(z)}{\partial z}\frac{\partial z}{\partial x}\Phi(x)\hat{\theta}\parallel. \tag{36}$$

It then follows from (31) that

$$\dot{\hat{V}} \leq -\delta\parallel z\parallel^2, \tag{37}$$

which is negative semi-definite.

It can be indicated from (20) and (37) that $\hat{\theta} \in \mathcal{L}^\infty$ and $z \in \mathcal{L}^\infty$, and $\hat{V}$ is (un-strictly) decreasing. Integrating (37) from $t = 0$ to $t = +\infty$ yields that

$$\int_0^{+\infty}\parallel z\parallel^2\,\mathrm{d}t \leq \frac{1}{\delta}\left(\hat{V}(0) - \hat{V}(+\infty)\right) \tag{38}$$

implying that $z \in \mathcal{L}^2$.

The boundedness of z and the continuous differentiability of f, g and κ indicate that system states and input are bounded; therefore $\dot{z}$ (or $\dot{x}$) is bounded, indicating that z is uniformly continuous.

Consequently, by using Barbalat lemma [21], it can be proved that $z \to 0$ as $t \to +\infty$. Since the $z = H(x)$ is a diffeomorphism, and $\hat{V}$ is radially unbounded, it can be claimed that the closed-loop system is globally asymptotically stable.

□

Remark 5: In the proposed adaptive MPC, although the cost function (11) [or stage cost function (22)] is not used explicitly for the proof of feasibility and stability, it is considerably important in affecting the transient performance of the closed-loop system.

4 Extension to adaptive MPC with soft constraints

The main disadvantage of the adaptive MPC proposed in Section 3 is that it cannot explicitly handle hard constraints expressed as $u \in \mathcal{U}$ or $x \in \mathcal{X}$. However, the proposed adaptive MPC can be extended to handle soft constraints.

In this section, we aim to constrain the states and input within the target constraints given by

$$\parallel x\parallel \leq \bar{x}, \quad \parallel u\parallel \leq \bar{u}, \tag{39}$$

where $\bar{x}$ and $\bar{u}$ are positive constants.

The concern is that, constraints (39) can be violated, but the violations would be severely penalised. To this end, the stage cost function for adaptive MPC with soft constraints can be designed by

$$\begin{aligned}l(x,u) = \parallel x\parallel_P + \parallel u\parallel_Q &+ \lambda_1 e^{\lambda_2(\parallel x\parallel - \bar{x} + \epsilon_1)}\\ &+ \lambda_3 e^{\lambda_4(\parallel u\parallel - \bar{u} + \epsilon_2)},\end{aligned} \tag{40}$$

where $\lambda_i\,(i = 1, 2, 3, 4)$ are positive weight constants; $\epsilon_i\,(i = 1, 2)$ are slack parameters indicating that the stage cost would increase when the states and input approach their constraints.

The implication of the stage cost function (40) is that, if the constraints given by (39) are violated, the stage cost would increase exponentially. Consequently, due to the optimisation with the stage cost function (40), the constraints (39) would be violated as little as possible.

Remark 6: Since the stage cost function does not appear explicitly in the proof of feasibility and stability, it can be appropriately modified according to some particular requirements without diminishing the results of Theorem 1.

5 Simulation examples

In this section, two simulation examples are presented to illustrate performances of the proposed MPC.

5.1 Example 1: First-order non-linear system

The plant to be controlled is given by a first-order system with one uncertain parameter

$$\dot{x} = x^3 + u + \theta x^2, \tag{41}$$

where $\theta = 0.8$ is the uncertain constant parameter and $\Phi(x) = x^2$. It is known that a conservative bound for the uncertain θ is given by $\bar{\theta} = 1.5$. Apparently, the first-order non-linear affine system (41) naturally satisfies Assumption 1. The nominal system

$$\dot{x} = x^3 + u \tag{42}$$

can be stabilised by

$$u = \kappa(x) = -x^3 - kx, \tag{43}$$

where $k > 0$ is the control gain.

Apparently, $V = (1/2)x^2$ is a feasible Lyapunov function for (42). According to (18), the adaptive updating law can be designed by

$$\dot{\hat{\theta}} = \mathrm{Prj}_{\hat{\theta}}(\gamma x^3), \tag{44}$$

where $\gamma = 0.4$ is assigned. The time-varying $\alpha_3(t)$ is obtained by solving the optimisation problem (29) subject to (31) with $\delta = 0.1$. The expected constraint for the control input is assigned by $\parallel u\parallel \leq 10$. In the MPC design, the predictive (control) horizon is assigned to $T = 0.8$ s. The initial value of the estimated parameter is set to $\hat{\theta}(0) = 0.5$, and the control parameter. The stage cost function is designed by (40), where the parameters are given by $P = 1$, $Q = 0.05$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 0.02$, $\lambda_4 = 2$, $\epsilon_1 = 0$, and $\epsilon_2 = 0.5$. Adaptive MPC is designed according to Algorithm 1 in Section 3.2. Suppose that initial value of the system state is $x(0) = 1.7$. Results are obtained by implementing Algorithm 1, and are displayed in Figs. 1–3.

As can be seen from Fig. 1, the state of the closed-loop system can be stabilised asymptotically by the proposed adaptive MPC. In Fig. 2, it can be seen that the control input violates its expected constraint slightly at its early stage; the reason is that the expected constraint is soft. Since the violation would be penalised exponentially, the control input then evolves within its constraint. Fig. 3 shows that, although it does not necessarily converge to the

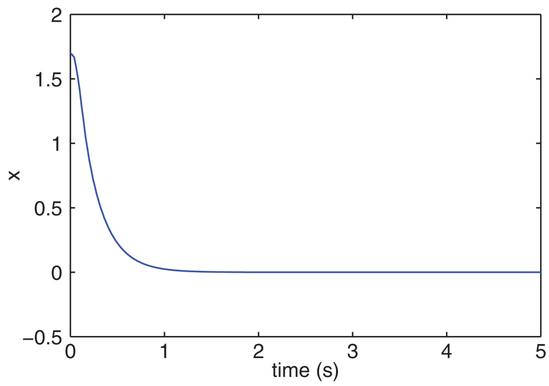

Fig. 1 *States of the closed-loop system in example 1*

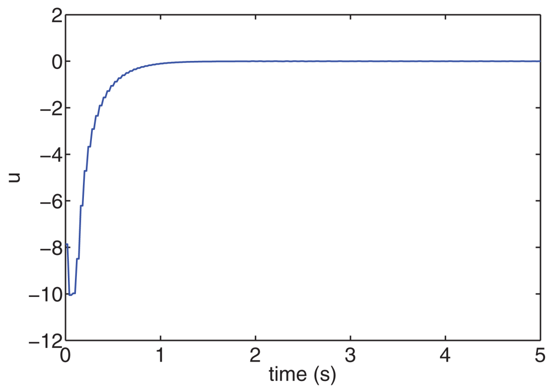

Fig. 2 *Control input of the closed-loop system in example 1*

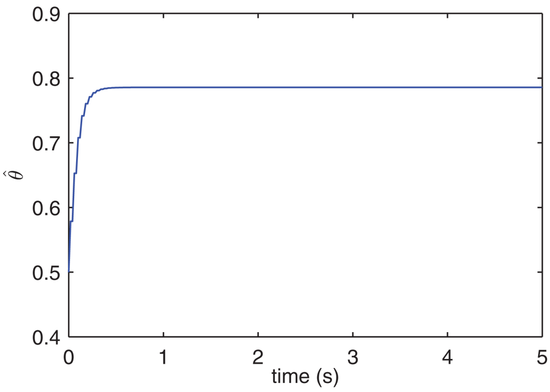

Fig. 3 *Adaptive estimated parameter in example 1*

real value of the corresponding uncertain parameter, the estimated parameter is bounded.

5.2 Example 2: Higher-order non-linear system

The plant to be controlled is given by a third-order system with two uncertain parameters:

$$\begin{cases} \dot{x}_1 & = x_1^2 - x_1^3 + x_2 + \theta_1 x_1, \\ \dot{x}_2 & = x_3 + \theta_1 x_1 + \theta_2 x_2, \\ \dot{x}_3 & = u, \end{cases} \tag{45}$$

where $\Theta = [\theta_1, \theta_2]^{\mathrm{T}} = [0.3, 0.7]^{\mathrm{T}}$ are uncertain constant parameters. A conservative bound for Θ can be given by $\bar{\theta} = 2$.

It can be seen from (45) that

$$\Phi(x) = \begin{bmatrix} x_1 & 0 \\ x_1 & x_2 \\ 0 & 0 \end{bmatrix}. \tag{46}$$

It is simple to calculate that the distribution

$$\operatorname{span}\{g, \mathrm{ad}_f g, \mathrm{ad}_f^2 g\} \tag{47}$$

is involutive and of rank 3, indicating that there exist a diffeomorphism and a feedback control, such that the nominal system

$$\begin{cases} \dot{x}_1 & = x_1^2 - x_1^3 + x_2, \\ \dot{x}_2 & = x_3, \\ \dot{x}_3 & = u, \end{cases} \tag{48}$$

can be transformed into (5). In fact, expressions for the diffeomorphism can be given by

$$\begin{cases} z_1 = x_1, \\ z_2 = x_1^2 - x_1^3 + x_2, \\ z_3 = (2x_1 - 3x_1^2)(x_1^2 - x_1^3 + x_2) + x_3, \end{cases} \tag{49}$$

and the feedback control $u = \alpha(x) + \beta(x)v$, where

$$\begin{aligned} \alpha(x) &= -(15x_1^4 - 20x_1^3 + 6x_1^2 - 6x_1x_2 + 2x_2)(x_1^2 - x_1^3 + x_2) \\ &\quad + (3x_1^2 - 2x_1)x_3, \\ \beta(x) &= 1. \end{aligned} \tag{50}$$

It follows that the eigenvalues of the closed-loop nominal system can be assigned arbitrarily by:

$$v = \kappa(z) = -k_1z_1 - k_2z_2 - k_3z_3, \tag{51}$$

where $k_i\,(i = 1, 2, 3)$ are control gains. If we set $k_1 = 8$, $k_2 = 12$, $k_3 = 6$, and select the Lyapunov function $V(z) = z^{\mathrm{T}}\Gamma z$, where

$$\Gamma = \begin{bmatrix} 1.9063 & 1.2344 & 0.0625 \\ 1.2344 & 2.0938 & 0.1445 \\ 0.0625 & 0.1445 & 0.1074 \end{bmatrix}, \tag{52}$$

then, along the solution of the nominal system (48), it holds that $\dot{V}(z) = -z^{\mathrm{T}}z$. It can also be calculated that $\alpha_4 \simeq 3.2$.

Remark 7: It should be noted that k_1, k_2, and k_3 do not appear explicitly in the proposed adaptive MPC; the purpose of selecting $k_1 = 8$, $k_2 = 12$, and $k_3 = 6$ is to find a feasible Lyapunov function $V(z)$ and its corresponding α_4.

In this example, the expected constraint for the control input is assigned by $\| u \| \le 2$. Correspondingly, the stage cost can be designed by (40) with parameters given by $P = \mathrm{diag}(1, 0.4, 0.04)$, $Q = 0.005$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 0.002$, $\lambda_4 = 8$, $\epsilon_1 = 0$, and $\epsilon_2 = 0.1$. The predictive (control) horizon is assigned to $T = 0.8$ s. Initial values of estimated parameters are set to $\hat{\Theta}(0) = [0.5, 0.5]^{\mathrm{T}}$. According to (18), the updating law for estimated parameters can be designed by

$$\dot{\hat{\Theta}} = \mathrm{Prj}_{\hat{\theta}}\big(\gamma\Phi(x)^{\mathrm{T}}\Gamma z\big), \tag{53}$$

where $\gamma = 0.4$ is assigned. The time-varying $\alpha_3(t)$ is obtained by solving the optimisation problem (29) subject to (31) with $\delta = 0.1$. Suppose that initial values of the system states are $x(0) = [1.1, -0.4, -0.1]^{\mathrm{T}}$. Results are obtained by implementing Algorithm 1 [with the stage cost function designed by (40)], and are displayed in Figs. 4–6.

As illustrated by Fig. 4, the states of the closed-loop system can be stabilised asymptotically by the proposed adaptive non-linear MPC. It can be seen in Fig. 5 that, with the proposed stage cost function (40) (that considers soft constraints), the control input violates the expected constraints slightly at the initial stage; the control input is capable of entering the constraints after very short transient process, since the violation is penalised exponentially by using (40). It is indicated by Fig. 6 that, although they do not

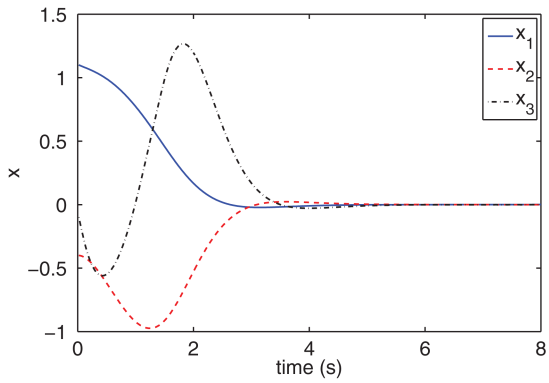

Fig. 4 *States of the closed-loop system in example 2*

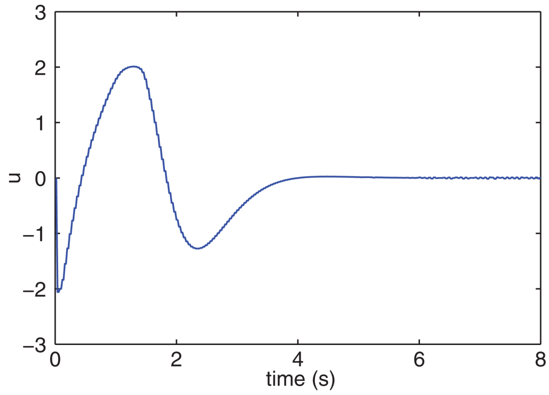

Fig. 5 *Control input of the closed-loop system in example 2*

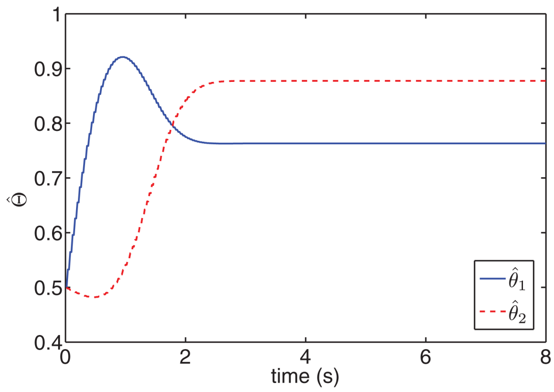

Fig. 6 *Adaptive estimated parameters in example 2*

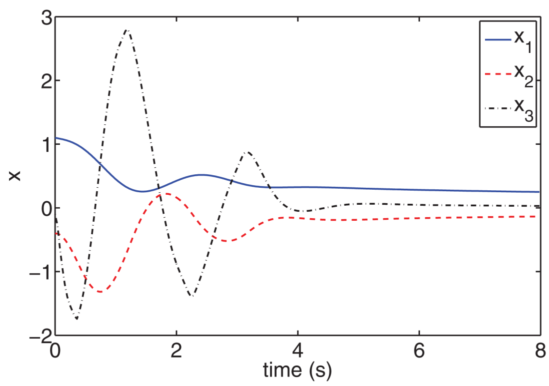

Fig. 7 *States of the closed-loop system with classic NMPC*

necessarily converge to the real values of the corresponding uncertain parameters, the estimated parameters are bounded.

To better evaluate superior performances and computational efficiency of the proposed adaptive MPC, classic non-linear MPC with terminal inequality constraint is applied to stabilised the non-linear system (48). The control horizon is assigned the same value as that in the proposed adaptive MPC. It is assumed that, in the classic MPC, the exact value of Θ is known, such that the prediction can be performed. The terminal inequality constraint is designed by $\| x \| < 0.6$ which is fairly large, such that optimisation can be feasible at initial stages. In the simulation, the optimisation tool is applied with algorithm 'interior-point' [23]. The results of classical MPC is displayed in Fig. 7, where states of the closed-loop system are capable of converging into small neighbourhood of zeros. They do not necessarily converge asymptotically to zeros; the reason is that, to improve feasibility and computational efficiency, terminal inequality constraint (instead of terminal equality constraint which guarantees the convergence to zeros) is applied. As can be seen from Fig. 7, the overshots are significantly larger than those in Fig. 4. Computational efficiencies of both the proposed adaptive MPC and the classic MPC are illustrated by Fig. 8, where a number of optimisation iterations are displayed. It can be seen from Fig. 8 that, compared with that of the classic MPC with terminal inequality constraint, number of iterations to perform the optimisation of the proposed adaptive MPC are fairly reduced during transient process, implying that the computational efficiency of the proposed adaptive MPC is significantly superior.

Remark 8: Number of optimisation iterations of the classic MPC are slightly smaller during steady-state; the reason is that, during steady-state, states of the closed-loopy system with the classic MPC has already converged into the inequality constraint, and it can be regarded that the optimisation is then processed without constraints.

6 Conclusion

In this paper, an adaptive model predictive control is proposed for a class of non-linear systems with constant parametric uncertainties. The control objective is to stabilise system states. Parametric uncertainties are estimated online by adaptive estimated parameters with a simple adaptive updating law, such that an estimated system can be constructed and used to predict future states. MPC is then designed with Lyapunov-based constraint, such that stability of the closed-loop system can be guaranteed, and computational burden of terminal penalties is reduced. Both theoretical proofs and simulation results demonstrate that, with the proposed adaptive MPC, states of the closed-loop system are stabilised asymptotically, and the estimated parameters are bounded.

Future works on this research may include: (i) analysis on sensitivity of the closed-loop system with respect to control parameters and (ii) extension of the proposed adaptive non-linear MPC to more general non-linear systems.

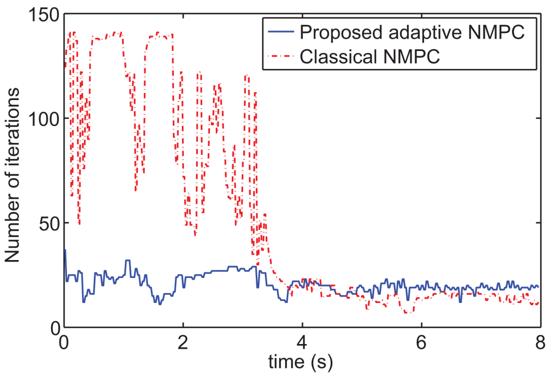

Fig. 8 *Number of iterations in optimisation*

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7 Appendix

7.1 Proof of (20)

The proof follows from the steps in [22]:

If $\| \hat\theta \| < \bar\theta$, the boundedness of $\hat\theta$ is satisfied.

If $\| \hat\theta \| = \bar\theta$ and $\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)(\partial V(x)/\partial x)^{\mathrm T} < 0$, select a Lyapunov function $V_\theta = (1/2)\hat\theta^{\mathrm T}\hat\theta$. Its derivative can be calculated by

$$\dot V_\theta = \hat\theta^{\mathrm T}\dot{\hat\theta} = \gamma\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm T} < 0, \tag{54}$$

implying that $\| \hat\theta \|$ decreases.

If $\| \hat\theta \| = \bar\theta$ and $\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)(\partial V(x)/\partial x)^{\mathrm T} \geq 0$, the derivative of V_θ can be calculated by

$$\begin{aligned}\dot V_\theta &= \hat\theta^{\mathrm T}\dot{\hat\theta} = \gamma\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm T} - \gamma\frac{\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm T}}{\| \hat\theta \|^2}\hat\theta^{\mathrm T}\hat\theta \\ &= 0,\end{aligned} \tag{55}$$

indicating that $\| \hat\theta \|$ is non-increasing. The second line of the above equation holds because the term $\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)(\partial V(x)/\partial x)^{\mathrm T}$ is a scalar.

In summary, if the initial value of $\hat\theta$ satisfies that $\| \hat\theta(0) \| < \bar\theta$, then $\| \hat\theta(t) \| < \bar\theta$ holds for all $t > 0$.

7.2 Proof of (21)

In the first two cases of (19),

$$\dot{\hat\theta} = \gamma\Phi^{\mathrm T}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm T}, \tag{56}$$

and (21) holds obviously.

In the third case of (19),

$$\dot{\hat\theta} = \gamma\Phi^{\mathrm T}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm T} - \gamma\frac{\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)(\partial V(x)/\partial x)^{\mathrm T}}{\| \hat\theta \|^2}\hat\theta, \tag{57}$$

and it follows that:

$$\begin{aligned}\dot V &= \frac{\partial V(x)}{\partial x}\left(f(x) + g(x)u + \Phi(x)\hat\theta\right) \\ &+ \gamma\frac{\hat\theta^{\mathrm T}\Phi^{\mathrm T}(x)(\partial V(x)/\partial x)^{\mathrm T}}{\| \hat\theta \|^2}\tilde\theta^{\mathrm T}\hat\theta,\end{aligned} \tag{58}$$

where

$$\Phi^{\mathrm T}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm T} \geq 0, \tag{59}$$

and

$$\begin{aligned}\tilde\theta^{\mathrm T}\hat\theta &= \theta^{\mathrm T}\hat\theta - \| \hat\theta \|^2 = \frac{1}{2}\| \theta \|^2 - \frac{1}{2}\| \theta - \hat\theta \|^2 - \frac{1}{2}\| \hat\theta \| \\ &\leq \frac{1}{2}\| \theta \|^2 - \frac{1}{2}\bar\theta^2 \leq 0.\end{aligned} \tag{60}$$

Consequently, (21) holds in this case.