Atomic Spin Polarization Controllability Analysis: A Novel Controllability Determination Method for Spin-Exchange Relaxation-Free Co-Magnetometers

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***Abstract*—This paper investigates the atomic spin polarization controllability of spin-exchange relaxation-free co-magnetometers (SERFCMs). This is the first work in the field of controllability analysis for the atomic spin ensembles systems, whose dynamic behaviors of spin polarization are described by the Bloch equations. Based on the Bloch equations, a state-space model of the atomic spin polarization for SERFCM is first established, which belongs to a particular class of nonlinear systems. For this class of nonlinear systems, a novel determination method for the global state controllability is proposed and proved. Then, this method is implemented in the process of controllability analysis on the atomic spin polarization of an actual SERFCM. Moreover, a theoretically feasible and reasonable solution of the control input is proposed under some physical constraints, with whose limitation of realistic conditions, the controller design can be accomplished more practically and more exactly. Finally, the simulation results demonstrate the feasibility and validation of the proposed controllability determination method.**

***Index Terms*—Atomic spin polarization, Bloch equations, controllability determination, nonlinear state-space model, spin-exchange relaxation-free co-magnetometer (SERFCM).**

I. Introduction

QUANTUM information technology has been the new main trend of information science and technology under the promotion of quantum mechanics and information technologies since the beginning of the 21st century [1]–[4]. Quantum precision measurement is one of the research directions closest to practical application as an important branch and research hotspot in the field of quantum information technology [5], [6]. The precision measuring instruments based on effects of atomic spin have become various types of indispensable equipments such as atomic clocks [7], atomic magnetometers [8], and atomic co-magnetometers [9], etc., for exploration on the leading science research frontiers by utilizing the characteristics of fine structures and quantization of transitions of atomic energy levels, where atomic co-magnetometer can be divided into two categories: the nuclear magnetic resonance gyroscopes (NMRG) [10] and the spin-exchange relaxation-free co-magnetometers (SERFCM) [11] according to their different working principles.

Alkali metal atoms within the cell of SERFCM may reach the SERF regime by the synthetic effects of high temperature and pump beam, in which the effect of polarization from spin-exchange rate is far greater than Larmor precession's, for hyperpolarizing the noble gas atoms. In the SERF regime, the signal intensity and signal-to-noise ratio are enhanced significantly as the improvements of atomic spin relaxation times, coherence and density. Furthermore, there is an interaction between electron and nuclear spin that can trace and compensate the slight fluctuation of the ambient magnetic field such that the high sensitivity for rotation sensing can be achieved in the SERF regime. Thus, SERFCM can not only be applied in inertial navigation as a promising gyroscope with high-performance [12], [13] but also serve for exploration in fundamental physics, for example the test of CPT and Lorentz violation [14]–[17], as well as searches for spin-dependent forces [18], [19].

The spherical cell almost filled with alkali metal vapors as well as the noble gas is a key component used in SERFCM for rotation sensing, and the dynamical evolution of spin polarization of these atoms within the cell can be described by a set of partial differential equations (PDEs) called the Bloch equations [17], which play an important role for interpreting the principles and characteristics of SERFCM. However, it is hard to obtain the complete analytical solution with respect to the time of such complicated PDEs. In this case, besides a

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series of theoretical and dynamic analyses [20]–[22], relevant studies put the main focus on practical engineering and always research the suppression or optimization for some kinds of perturbation and performance indexes based on the steady-state solution. For instance, the studies of magnetic noise-induced rotation measurement error [23], cross-talk effect in the dual-axis co-magnetometers [24], fluctuation of pump-beam intensity [25] and influence of the cell temperature [26] are always hotpots. It is worth noting that there are scholars who recently have introduced the data-driven state observation (DDSO) method for estimating the nuclear spin polarization in real time [27]. We believe that the organic combination of control theory and SERFCM is one of the unneglected future directions but underestimated now, and this work above exactly demonstrates the potential for this kind of combination.

So far, there has been few works focusing on the studies of SERFCMs from the perspective of control theory, which were concentrated on stability analysis, like long-term stability [28], [29] and bias stability [30], [31], and the studies on properties of SERFCMs involving controllability and observability in control domain were rarely reported, especially lacking on issues related to controllability. The concept of controllability emerged with the proposition of the state-space method in the 1960s for studying whether the systems can be controlled effectively or not by control input [32]. In general, the Bloch equations can be applied directly as the state-space model of SERFCMs in which the state vector contains the polarization of atomic spin ensembles. Nevertheless, the existing controllability determination method is not feasible for this system model of SERFCM because of its high nonlinearity and unqualified system features [33]–[36]. Under this situation, there are two distinct approaches: one way is to simplify the nonlinear model to a linear system directly for reducing the analysis difficulty, but excessive elimination of nonlinear terms may cause a startling deviation between the results we derived and reality; another way is to appropriately simplify the Bloch equations to the nonlinear system with a specific form or any other characteristics, then design the corresponding controllability analysis methods from its form and characteristics. Actually, the new controllability determination method proposed in this paper is exactly based on the second approach.

In the research field of SERFCM, there is no report on the atomic spin polarization controllability, which follows that the theoretical results in this paper are pioneering and groundbreaking. Furthermore, though the study object in this paper is SERFCM, the research idea on the controllability of the atomic spin ensembles systems, whose dynamic behaviors are described by the corresponding Bloch equations, has genuine generality. Therefore, our conclusions have certain universality and potential to be expanded to other precision measuring instruments, whose working principles are based on the effects and dynamical evolutions of the atomic spin described by the Bloch equations.

The structure of this paper is as follows. Section II derives the atomic spin polarization model from the Bloch equations, which is a nonlinear system in nature with a specific form. For this class of nonlinear systems, Section III proposes and proves the criterion condition of global state controllability. Section IV demonstrates the controllability analysis process for an actual SERFCM. Section V verifies the validation of our new controllability determination method by the numerical simulation of the K-^{3}He SERFCM. Finally, conclusions are drawn in Section VI.

II. Atomic Spin Polarization Model of SERFCM and Its Controllability Problem

In the working state of SERFCM, the atomic spin polarization is composed of the electron polarization $\vec{P}^e$ of alkali metal vapors and the nuclear polarization $\vec{P}^n$ of noble gas, whose dynamic behaviors can be described by the following Bloch equations:

$$\frac{\partial \vec{P}^e}{\partial t} = \frac{\gamma_e}{Q}\left(\vec{B}+\lambda M_n\vec{P}^n+\vec{L}\right)\times\vec{P}^e-\vec{\Omega}\times\vec{P}^e + \frac{1}{Q}\left(R_p\vec{S}_p+R_m\vec{S}_m+R_{se}^{en}\vec{P}^n-R_{tot}^{e}\vec{P}^e\right) \quad (1)$$

$$\frac{\partial \vec{P}^n}{\partial t} = \gamma_n\left(\vec{B}+\lambda M_e\vec{P}^e\right)\times\vec{P}^n-\vec{\Omega}\times\vec{P}^n + R_{se}^{ne}\vec{P}^e-R_{tot}^{n}\vec{P}^n. \quad (2)$$

Here γ_e and γ_n are the electron and nuclear gyromagnetic ratio, $\vec{B}$ is the ambient magnetic field vector, $\lambda M_e\vec{P}^e$ and $\lambda M_n\vec{P}^n$ are magnetic fields generated by the magnetization of the electron and nuclear spins, respectively. $\vec{\Omega}$ is the input rotation rate vector. Q is the slowing-down factor reflecting the ability that can slow down all of the rates for electron polarization, that is generated by the interactions between electron and nuclear spins [37]. $\vec{L}$ is the light-shift affecting electron spin as a magnetic field. R_p and R_m are the pumping rates from pump beam and probe beam, $\vec{S}_p$ and $\vec{S}_m$ give the corresponding directions of these laser beams above. R_{se}^{ne} and R_{se}^{en} are the spin-exchange rates produced by electrons and nuclei. R_{tot}^{e} and R_{tot}^{n} are the total relaxation rates of electron and nuclear spins, respectively. In this paper, the directions of pump laser and probe laser are defined to be *z*- and *x*-axes, respectively. In addition, the longitudinal direction is always defined as the direction along *z*-axis, and the transverse directions are defined as the directions along *x*- and *y*-axes.

According to the Bloch equations, magnetic field and rotation rate can be experienced by electron and nuclear spin ensembles. In the condition of extremely weak transverse magnetic field (less than 1 nT) and high temperature (about 470 K), the electron spins of alkali metal are rapidly polarized by a circularly polarized pump beam, and the nuclear spins of noble gas are uninterruptedly polarized by those alkali metal atoms who have the macroscopical spin direction for several hours. Simultaneously, the electron spins evolution is affected by a certain magnetic field $\lambda M_e\vec{P}^n$ and the pumping rate $R_{se}^{en}\vec{P}^n$, which are generated by nuclear spins. To some extent, the nuclear spins function as a feedback part in the dynamical evolution of the atomic spin ensembles. Eventually, under the joint effects of the magnetic field, rotation rate, polarization and relaxation, the polarization of the electron and nuclear spin ensembles reach an equilibrium point that is the notion of pumping steady-state (PSS) in this paper, and SERFCM can achieve sensitive rotation sensing in this steady-state.

Previous studies [17], [28] derived the approximate steady-state solutions by solving $\partial P_z^e/\partial t = \partial P_z^n/\partial t = 0$.

$$\bar{P}_z^e = \frac{R_p}{R_{tot}^e} \tag{3}$$

$$\bar{P}_z^n = \frac{R_{se}^{ne}}{R_{tot}^n}\bar{P}_z^e = \frac{R_{se}^{ne}R_p}{R_{tot}^n R_{tot}^e}. \tag{4}$$

However, as an actual working instrument for rotation sensing, the input rotation rate $\vec{\Omega}$ of SERFCM can not be referred as a known constant value such that there exists the variation of longitudinal polarization even at PSS. Thus we consider redefining PSS to be a approximate steady-state with minor fluctuation, which is weaker than $\partial P_z^e/\partial t = \partial P_z^n/\partial t = 0$ but closer to realistic condition.

$$\begin{cases} \sup\limits_t \left\| \dfrac{\partial P_z^e}{\partial t} \right\| < \delta_e \\ \sup\limits_t \left\| \dfrac{\partial P_z^n}{\partial t} \right\| < \delta_n \end{cases} \tag{5}$$

where the thresholds δ_e and δ_n can be exploited to judge if SERFCM is at PSS as the criterion condition. In this paper, the implementation of any controlling measure is based on a basic assumption that SERFCM is under PSS we defined.

The longitudinal magnetic field $B_z = -\lambda M_e \bar{P}_z^e - \lambda M_n \bar{P}_z^n$ is always employed as the compensation point for obtaining higher rotation measurement sensitivity. The polarization-dependent slowing-down factor Q is assumed to be a constant. The pumping rate R_m from probe laser can be neglected because of $R_m \ll R_p$. In addition, the light-shift generated by electrons needs to be eliminated by some technical means [38]. Under these conditions above, the Bloch equations (1) and (2) can be rewritten as a nonlinear system with the specific form as follows:

$$\dot{x} = \mathcal{F}(x) + \mathcal{H}u. \tag{6}$$

Here, $x = \left[P_x^e\ P_y^e\ P_z^e\ P_x^n\ P_y^n\ P_z^n\right]^{\mathrm{T}} \in \mathbb{R}^6$ is the state vector contains transverse and longitudinal polarization components of atomic spin ensembles, $u = \left[B_x\ B_y\ \Omega_x\ \Omega_y\right]^{\mathrm{T}} \in \mathbb{R}^4$ containing the transverse components of magnetic field and rotation rate, the mapping $\mathcal{F} : \mathbb{R}^6 \to \mathbb{R}^6$ is a nonlinear function.

The constant matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{bmatrix} 0 & \gamma_e P_z^e/Q & 0 & -P_z^e \\ -\gamma_e P_z^e/Q & 0 & P_z^e & 0 \\ \gamma_e P_y^e/Q & -\gamma_e P_x^e/Q & -P_y^e & P_x^e \\ 0 & \gamma_n P_z^n & 0 & -P_z^n \\ -\gamma_n P_z^n & 0 & P_z^n & 0 \\ \gamma_n P_y^n & -\gamma_n P_x^n & -P_y^n & P_x^n \end{bmatrix}. \tag{8}$$

It should be noted that the longitudinal electron and nuclear spin polarization are far greater than their transverse polarization components, which means $P_z^e \gg P_{x/y}^e$ and $P_z^n \gg P_{x/y}^n$. Besides, the conditions in (5) of PSS have indicated that the variations of the longitudinal polarizations P_z^e and P_z^n are very little. Moreover, the coefficients γ_e, γ_n and Q are constants. In this case, $\mathcal{H}$ can be regarded as a constant matrix in the normal working state of SERFCM. For detailed justification of this simplification, please see Appendix.

Remark 1: In system (6), the electron polarization of alkali metal vapors and nuclear polarization of noble gas are directly affected by the nonlinear function $\mathcal{F}(x)$ and the linear term $\mathcal{H}u$. For estimating the longitudinal electron and nuclear spin polarization while SERFCM is at PSS, we can suppose that $P_x^{e/n} = P_y^{e/n} = 0$, and then the solutions of differential equation $\partial P_z^e/\partial t = \partial P_z^n/\partial t = 0$ are obtained in the absence of effect from $\mathcal{H}u$.

$$\bar{P}_z^e = \frac{R_p + R_{se}^{en}\bar{P}_z^n}{R_{tot}^e} = \frac{R_p R_{tot}^n}{R_{tot}^e R_{tot}^n - R_{se}^{en} R_{se}^{ne}} \tag{9}$$

$$\bar{P}_z^n = \frac{R_{se}^{ne}}{R_{tot}^n}\bar{P}_z^e = \frac{R_{se}^{ne} R_p}{R_{tot}^e R_{tot}^n - R_{se}^{en} R_{se}^{ne}}. \tag{10}$$

These solutions will agree well with previous studies (3) and (4) if we further simplify (9) and (10) by neglecting small terms, it follows that the model we established is applicable for SERFCM. Moreover, the precision of estimation results for longitudinal electron and nuclear spin polarization are improved explicitly in comparison with previous studies.

The electron polarization of alkali metal and the nuclear polarization of noble gas play a significant role in the establishment of the theoretical model of SERFCM and they can describe the performance explicitly. The nature of the atomic spin polarization model (6) of SERFCM is a nonlinear system with a specific form and it is necessary for us to analyze its controllability theoretically.

$$\mathcal{F}(x) = \begin{bmatrix} \gamma_e \lambda M_n (P_z^e P_y^n - P_y^e P_z^n)/Q + (R_{se}^{en} P_x^n - R_{tot}^e P_x^e)/Q - \gamma_e P_y^e B_z/Q \\ \gamma_e \lambda M_n (P_x^e P_z^n - P_z^e P_x^n)/Q + (R_{se}^{en} P_y^n - R_{tot}^e P_y^e)/Q + \gamma_e P_x^e B_z/Q \\ \gamma_e \lambda M_n (P_y^e P_x^n - P_x^e P_y^n)/Q + (R_{se}^{en} P_z^n - R_{tot}^e P_z^e)/Q + R_p/Q \\ \gamma_n \lambda M_e (P_y^e P_z^n - P_z^e P_y^n) + (R_{se}^{ne} P_x^e - R_{tot}^n P_x^n) - \gamma_n P_y^n B_z \\ \gamma_n \lambda M_e (P_z^e P_x^n - P_x^e P_z^n) + (R_{se}^{ne} P_y^e - R_{tot}^n P_y^n) + \gamma_n P_x^n B_z \\ \gamma_n \lambda M_e (P_x^e P_y^n - P_y^e P_x^n) + (R_{se}^{ne} P_z^e - R_{tot}^n P_z^n) \end{bmatrix}. \tag{7}$$

III. Proposed Controllability Determination Method

In this section, the criterion condition for controllability and corresponding proof are proposed for analyzing the nonlinear system (6).

A. Preliminaries

Throughout this section, all the matrices are defined in real number field and the following mathematical notations are used. $\lambda_i(A)$, for $A\in\mathbb{R}^{n\times n}$, denotes the i-th eigenvalue of matrix A. $\mathrm{Re}(\lambda)$ denotes the real part of complex number $\lambda\in\mathbb{C}$. For $A\in\mathbb{R}^{n\times n}$, $\mathrm{Rank}(A)$ is the rank of matrix A. $\ker(A)$, for $A\in\mathbb{R}^{m\times n}$, denotes the kernel of matrix A and can be defined as $\ker(A)=\{x\in\mathbb{R}^n|Ax=0\}$.

Let $t\geq 0$, the dynamic model for the continuous-time nonlinear systems with a specific form can be expressed as

$$\dot{x}(t)=\mathcal{F}[x(t)]+\mathcal{H}u(t) \tag{11}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ are the bounded state and bounded input vectors of system (11), respectively. The matrix $\mathcal{H}\in\mathbb{R}^{n\times m}$ is constant and the mapping $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}^n$ is a known nonlinear function.

Assumption 1: For nonlinear system (11), assume that

1) Function $\mathcal{F}[x(t)]$ has derivative at the origin $x(t)=0$, we define the matrix $\mathcal{A}=\frac{\partial\mathcal{F}}{\partial x}[x(t)]\big|_{x(t)=0}$;

2) For matrix $\mathcal{A}\in\mathbb{R}^{n\times n}$, $\max_i \mathrm{Re}[\lambda_i(\mathcal{A})]<0$, where $\lambda_i(1\leq i\leq n)$ are the eigenvalues of $\mathcal{A}$;

3) From Condition 1), define $\mathcal{G}[x(t)]=\mathcal{F}[x(t)]-\mathcal{A}x(t)$, where it is supposed that $\forall x\in\mathbb{R}^n$, $\|\mathcal{G}[x(t)]\|_2<\infty$.

Lemma 1 [39]: For all matrices $\mathcal{A}\in\mathbb{R}^{n\times n}$, if $\max\limits_i \mathrm{Re}[\lambda_i(\mathcal{A})]<0$, then $\exists\alpha>0$ and $s>0$ are both constants such that $\|e^{\mathcal{A}t}\|_2\leq\alpha e^{-st}$.

Lemma 2: $\forall\mathcal{A}\in\mathbb{R}^{n\times n}$, $\mathcal{H}\in\mathbb{R}^{n\times m}$, let matrix $S_c\in\mathbb{R}^{n\times nm}$ defined as $S_c=[\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^{n-1}\mathcal{H}]$. Then with the corresponding Gram matrix $W(0,t)=\int_0^t e^{-\mathcal{A}\tau}\mathcal{H}\mathcal{H}^{\mathrm{T}}e^{-\mathcal{A}^{\mathrm{T}}\tau}d\tau$, $t>0$, $\mathrm{Rank}(S_c)=n$ if and only if $\exists t>0$ such that $W(0,t)$ is nonsingular.

Proof: Construct a linear times-invariant (LTI) system ($\mathcal{A}$, $\mathcal{H}$), where $\mathcal{A}\in\mathbb{R}^{n\times n}$ and $\mathcal{H}\in\mathbb{R}^{n\times m}$. Then we can deduce that

1) The LTI system ($\mathcal{A}$, $\mathcal{H}$) is globally state-controllable if and only if $\mathrm{Rank}(S_c)=n$.

2) The LTI system ($\mathcal{A}$, $\mathcal{H}$) is globally state-controllable if and only if $\exists t>0$ such that $W(0,t)$ is nonsingular.

Therefore, $\mathrm{Rank}(S_c)=n$ if and only if $\exists t>0$ such that $W(0,t)$ is nonsingular. ■

B. The Controllability Determination Method for a Particular Class of Nonlinear Systems

Under Assumption 1, system (11) can be rewritten as follows:

$$\dot{x}(t)=\mathcal{A}x(t)+\mathcal{H}u(t)+\mathcal{G}[x(t)]. \tag{12}$$

Theorem 1: Under Assumption 1, these two equivalent nonlinear systems (11) and (12) are globally state-controllable if and only if $\mathrm{Rank}([\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^{n-1}\mathcal{H}])=n$.

Proof (Sufficiency): $\mathrm{Rank}([\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^{n-1}\mathcal{H}])=n$ is known.

We suppose that $x_0=x(0)\in\mathbb{R}^n$ is the initial state and $x_T=x(T)\in\mathbb{R}^n$ is the expected terminal state, where $T>0$.

The analytical solution of partial differential equation (12) is given as follows:

$$x(T)=e^{\mathcal{A}T}x_0+\int_0^T e^{\mathcal{A}(T-t)}\mathcal{G}[x(t)]dt + \int_0^T e^{\mathcal{A}(T-t)}\mathcal{H}u(t)dt. \tag{13}$$

We consider the boundness of $\int_0^T e^{\mathcal{A}(T-t)}\mathcal{G}[x(t)]dt$ at first. By Cauchy-Schwarz inequation,

$$\left\|\int_0^T e^{\mathcal{A}(T-t)}\mathcal{G}[x(t)]dt\right\|_2 \leq\left[\int_0^T \|e^{\mathcal{A}(T-t)}\|_2dt\right]^{\frac{1}{2}}\left[\int_0^T \|\mathcal{G}[x(t)]\|_2dt\right]^{\frac{1}{2}}. \tag{14}$$

Then by Lemma 1, we can introduce the parameter α, and it is noted that the magnitude of transition time T depends on $\alpha<0$, where the closer α is to zero, the longer transition time T is

$$\begin{aligned}\left\|\int_0^T e^{\mathcal{A}(T-t)}\mathcal{G}[x(t)]dt\right\|_2 &\leq\left[\int_0^T \alpha e^{-s(T-t)}dt\right]^{\frac{1}{2}}\left[\int_0^T \|\mathcal{G}[x(t)]\|_2dt\right]^{\frac{1}{2}}\\ &=\sqrt{\frac{1-e^{-sT}}{\alpha s}}\left[\int_0^T \|\mathcal{G}[x(t)]\|_2dt\right]^{\frac{1}{2}}<\infty.\end{aligned} \tag{15}$$

Thus, $\forall\mathcal{A}\in\mathbb{R}^{n\times n}$, $\int_0^T e^{\mathcal{A}(T-t)}\mathcal{G}[x(t)]dt$ is bounded. Finally, the corresponding Gram matrix $W(0,T)$ is nonsingular by Lemma 2, in that case an existent but non-exclusive control input is

$$u(t)=-\mathcal{H}^{\mathrm{T}}e^{-\mathcal{A}^{\mathrm{T}}t}W^{-1}(0,T)\times\left[x_0-e^{-\mathcal{A}T}x_T+\int_0^T e^{\mathcal{A}(T-t)}\mathcal{G}[x(t)]dt\right] \tag{16}$$

such that $x(T)=x_T$. Therefore, the state vector of system (12) can be transferred from any bounded initial state x_0 to any expected bounded terminal state x_T in the limited time $(0,T)$ by at least one specific control input $u(t)$, which means that system (12) is globally state-controllable. ■

Proof (Necessity): The global state controllability for system (12) is known.

We suppose that $\mathrm{Rank}([\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^{n-1}\mathcal{H}])<n$, then $\forall t>0$, the corresponding Gram matrix $W(0,t)$ is singular by Lemma 2. Hence there inevitably exists a non-zero vector $\beta\in\mathbb{R}^n$ such that

$$\beta^{\mathrm{T}}W(0,t)\beta=\int_0^t(\beta^{\mathrm{T}}e^{-\mathcal{A}\tau}\mathcal{H})(\beta^{\mathrm{T}}e^{-\mathcal{A}\tau}\mathcal{H})^{\mathrm{T}}d\tau=0 \tag{17}$$

which means

$$\forall\tau\in[0,t],\beta^{\mathrm{T}}e^{-\mathcal{A}\tau}\mathcal{H}=0. \tag{18}$$

Since the system (12) is globally state-controllable, let terminal state $x(T)=0$, then we have

$$x_0=-\int_0^T e^{-\mathcal{A}t}\mathcal{G}[x(t)]dt-\int_0^T e^{-\mathcal{A}t}\mathcal{H}u(t)dt. \tag{19}$$

From (18) and (19), we can deduce that

$$\begin{aligned}\beta^{\mathrm{T}} x_0 &= -\beta^{\mathrm{T}}\left[\int_0^T e^{-\mathcal{A}t}\mathcal{G}[x(t)]dt + \int_0^T e^{-\mathcal{A}t}\mathcal{H}u(t)dt\right]\\ &= -\int_0^T \beta^{\mathrm{T}} e^{-\mathcal{A}t}\mathcal{G}[x(t)]dt\\ &\Rightarrow \beta^{\mathrm{T}}\left[x_0 + \int_0^T e^{-\mathcal{A}t}\mathcal{G}[x(t)]dt\right] = 0. \end{aligned} \tag{20}$$

Let $\gamma = x_0 + \int_0^T e^{-\mathcal{A}t}\mathcal{G}[x(t)]dt$ in (20), obviously $\gamma \in \ker(\beta^{\mathrm{T}})$, where $\ker(\beta^{\mathrm{T}})$ is the subspace of $\mathbb{R}^n$ and the number of its dimensions are $(n-1)$. In that case, the value of initial state vector x_0 is also constrained into an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, which is contrary to the assumption that system (12) is global state-controllable.

Therefore, $\mathrm{Rank}([\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^{n-1}\mathcal{H}]) = n$. ■

Remark 2: Since the Bloch equations can be exploited to describe the dynamic behaviors of many atomic spin ensembles systems, the polarization controllability criterion proposed in Theorem 1 can be expanded to other quantum precision measurement instruments based on the atomic spin effect, such as the SERF magnetometer [8] and the NMR gyroscope [10], etc. For the quantum precision instruments that cannot conform to the Bloch equations, their state controllability also can be determined by Theorem 1 as long as their dynamic behaviors can satisfy the mathematical description of model (11) and Assumption 1. However, for the nonlinear systems with more general forms rather than model (11) or not satisfying Assumption 1, we cannot provide a universal state controllability criterion.

IV. Atomic Spin Polarization Controllability Analysis for SERFCM Using the Proposed Method

As we mentioned before, the Bloch equations can be rewritten as a nonlinear system with the specific form while SERFCM is considered at PSS. For this class of nonlinear systems, a novel controllability determination method is provided, which can be applied to analyze the atomic spin polarization controllability for SERFCM.

The validation of the controllability determination method we propose in this paper is based on Assumption 1, thus we need to verify whether system (6) satisfies Assumption 1 or not at first.

According to (7), it is not hard to find that nonlinear function $\mathcal{F}$ has derivative at the origin, then we differentiate $\mathcal{F}$ with respect to x

$$\mathcal{A} = \left.\frac{\partial\mathcal{F}}{\partial x}\right|_{x=0} = \begin{bmatrix} \frac{-R_{tot}^e}{Q} & \frac{-\gamma_e B_z}{Q} & 0 & \frac{R_{se}^{en}}{Q} & 0 & 0\\ \frac{\gamma_e B_z}{Q} & \frac{-R_{tot}^e}{Q} & 0 & 0 & \frac{R_{se}^{en}}{Q} & 0\\ 0 & 0 & \frac{-R_{tot}^e}{Q} & 0 & 0 & \frac{R_{se}^{en}}{Q}\\ R_{se}^{ne} & 0 & 0 & -R_{tot}^n & -\gamma_n B_z & 0\\ 0 & R_{se}^{ne} & 0 & \gamma_n B_z & -R_{tot}^n & 0\\ 0 & 0 & R_{se}^{ne} & 0 & 0 & -R_{tot}^n \end{bmatrix}. \tag{21}$$

Actually, the magnitude of spin-exchange rate R_{se}^{ne} produced by the electron is very low in the interactions between electron and nuclear spins, which implies the reason why we always need several hours to uninterruptedly hyperpolarize the noble gas in practical application. Hence the smaller term R_{se}^{ne} can be neglected, and $\mathcal{A}$ in (21) becomes an upper triangular matrix in that case, then the analytical solutions of eigenvalues of $\mathcal{A}$ can be calculated expediently.

$$\lambda(\mathcal{A}) = \{\lambda_1(\mathcal{A}),\ \lambda_2(\mathcal{A}), \ldots,\ \lambda_6(\mathcal{A})\} \tag{22}$$

where

$$\begin{aligned}\lambda_{1,2}(\mathcal{A}) &= -\frac{R_{tot}^e}{Q} \pm i\frac{\gamma_e B_z}{Q}\\ \lambda_{3,4}(\mathcal{A}) &= -R_{tot}^n \pm i\gamma_n B_z\\ \lambda_5(\mathcal{A}) &= -\frac{R_{tot}^e}{Q}\\ \lambda_6(\mathcal{A}) &= -R_{tot}^n \end{aligned} \tag{23}$$

and all the eigenvalues satisfy that $\mathrm{Re}[\lambda_i(\mathcal{A})] < 0$, $i \in \{1, 2, \ldots, 6\}$, thus we can obtain $\max_i \mathrm{Re}[\lambda_i(\mathcal{A})] < 0$.

Then consider the nonlinear term $\mathcal{G}$ by definition

$$\mathcal{G}(x) = \mathcal{F}(x) - \mathcal{A}x = \begin{bmatrix} \gamma_e\lambda M_n(P_z^e P_y^n - P_y^e P_z^n)/Q\\ \gamma_e\lambda M_n(P_x^e P_z^n - P_z^e P_x^n)/Q\\ \gamma_e\lambda M_n(P_y^e P_x^n - P_x^e P_y^n)/Q + R_p/Q\\ \gamma_n\lambda M_e(P_y^e P_z^n - P_z^e P_y^n)\\ \gamma_n\lambda M_e(P_z^e P_x^n - P_x^e P_z^n)\\ \gamma_n\lambda M_e(P_x^e P_y^n - P_y^e P_x^n) \end{bmatrix}. \tag{24}$$

Obviously, $\forall x \in \mathbb{R}^n$, $\|\mathcal{G}(x)\|_2 < \infty$.

According to (21)–(24), nonlinear system (6) derived from the Bloch equations explicitly satisfies the Assumption 1 such that the controllability determination method can be exploited to analyze its controllability.

In practical engineering, the atomic source of SERFCM is the key factor for sensing rotation and K-^{3}He is a kind of feasible combination.

When considering SERFCM based on K-^{3}He atomic source, $\mathrm{Rank}([\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^5\mathcal{H}]) = 6$ can be verified by computation with utilizing these typical values in Table I. In that case, system (6) is globally state-controllable such that polarization of K-^{3}He SERFCM can be fully controlled at PSS.

Remark 3: Theoretically, the analytical result of rank of matrix $S_c = [\mathcal{H}\ \mathcal{A}\mathcal{H}\ \cdots\ \mathcal{A}^5\mathcal{H}]$ may be calculated directly without substitution of those typical values in Table I. Since the number of columns is greater than that of rows in matrix $S_c \in \mathbb{R}^{6\times 24}$ as well as that $\mathcal{A}$ and $\mathcal{H}$ are not the sparse matrices, thus it is comparatively easy for S_c to reach row full rank. This suggest that the theoretical probability of achieving global state controllability for SERFCM based on kinds of atomic source schemes, for instance, Cs-^{129}Xe, K-Rb-^{21}Ne, Cs-Rb-^{21}Ne, etc., is comparatively high.

Consequently, with the further study around SERFCM, the atomic source schemes can be foreseen to involve more and complicated combinations, and the check for the full rank

TABLE I
TYPICAL VALUES [17] FOR SERFCM BASED ON K-^{3}HE ATOMIC SOURCE AT PSS, WHERE THE LONGITUDINAL POLARIZATIONS $\bar{P}_z^e$ AND $\bar{P}_z^n$ ARE CALCULATED BY (9) AND (10), RESPECTIVELY. THE POLARIZATION-DEPENDENT SLOWING-DOWN FACTOR IS CALCULATED BY DEFINITION. WE IGNORE THE LIGHT-SHIFT L AND THE MOST RELEVANT ROTATIONS ARE THOSE OF THE EARTH

Term	Typical value	Term	Typical value
γ_e	$2\pi\times 2.8\times 10^{10}$ Hz/T	γ_n	$2\pi\times 3.2\times 10^{7}$ Hz/T
R_{tot}^e	353 1/s	R_{tot}^n	2×10^{-4} 1/s
R_{se}^{en}	36 1/s	R_{se}^{ne}	4×10^{-6} 1/s
λM_e	-28×10^{-10} T	λM_n	-130×10^{-7} T
R_p	166 1/s	R_m	10^{-2} 1/s
$\bar{P}_z^e$	47.1216%	$\bar{P}_z^n$	0.09424%
Q	5.27	B_z	1.2383×10^{-9} T

property of S_c is necessary while the controllability determination method this paper provided is applied to the polarization controllability analysis for SERFCM based on kinds of atomic sources in practical engineering.

Remark 4: Throughout this paper, the achievement of global state controllability for SERFCM is based on a basic assumption that PSS is kept during the whole operation or control process, which can be interpreted from these perspectives below.

1) Considering the system (6) from the perspective of control theory, although the bounded initial state x_0 can be transferred to any bounded terminal state x_T in the state-space $\mathbb{R}^6$, the state trajectory $x(t)$, where $t\in[0,T]$, must satisfy those criterion conditions (5).

2) From another perspective, in practical engineering, as the high-sensitivity property of SERFCM for rotation and magnetic field sensing, the electron and nuclear spin polarization vary rapidly and significantly while the impacted excitations, like an unexpected magnetic field or high rotation rate, are employed, it is not until the absence of excitations that SERFCM can restore to PSS, which is always taking a long time.

Therefore, even if we have acquired the results of controllability for various SERFCMs, the magnitudes of $\|u(t)\|$ and $\|\partial P_z^{e/n}/\partial t\|$ from state trajectory still need to be considered carefully while we design the controller.

Remark 5: Besides SERFCM, there are many precision measuring instruments based on effects of atomic spin, like SERF magnetometer [40] and NMRG [41], their working principles can be described by different forms of Bloch equations, which implies that our research has potential to be generalized to theoretical models of them.

V. NUMERICAL SIMULATION

Considering an SERFCM based on K-^{3}He atomic source, the relevant typical values are shown in Table I, and their global state controllability has been analyzed and proved in Sections 3 and 4.

Set the initial state and the expected terminal state as follows:

$$\begin{aligned} x_0 &= [0\ 0\ \bar{P}_z^e\ 0\ 0\ \bar{P}_z^n]^{\mathrm{T}} \\ x_T &= [P_{x,T}^e\ P_{y,T}^e\ \bar{P}_{z,T}^e\ P_{x,T}^n\ P_{y,T}^n\ \bar{P}_{z,T}^n]^{\mathrm{T}} \end{aligned} \tag{25}$$

where

$$\begin{aligned} P_{x,T}^e &= 8\times 10^{-4} \\ P_{y,T}^e &= -4\times 10^{-3} \\ P_{x,T}^n &= 9\times 10^{-6} \\ P_{y,T}^n &= -5\times 10^{-6}. \end{aligned} \tag{26}$$

According to (5), a set of reasonable threshold that may be applied to practical engineering is given

$$\begin{aligned} \delta_e &= 10^{-5} \\ \delta_n &= 10^{-8}. \end{aligned} \tag{27}$$

Under these conditions, we can find at least one feasible specific control input as shown in Fig. 1, then Figs. 2 and 3 illustrate the variation of transverse polarization components $P_{x/y}^{e/n}$ as well as longitudinal polarization components $P_z^{e/n}$, respectively. Finally, Fig. 4 demonstrates the variation ratio of electron and nuclear spin polarization.

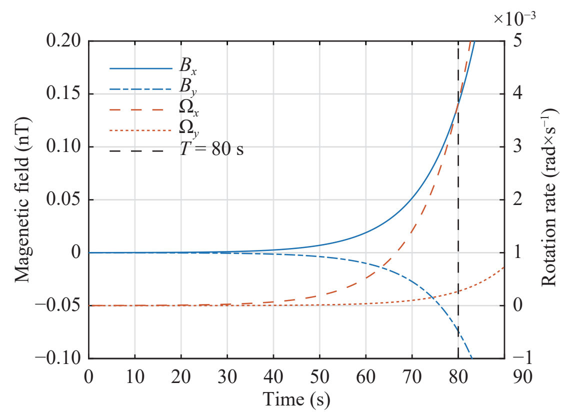

Fig. 1. A feasible but non-exclusive control input $u(t) = \left[B_x\ B_y\ \Omega_x\ \Omega_y\right]^{\mathrm{T}}$ for transferring the initial state x_0 to the expected terminal state x_T. In order to keep SERFCM system at PSS, (5) and (27) must be satisfied, such that the control input must be a slowly time-varying smooth function.

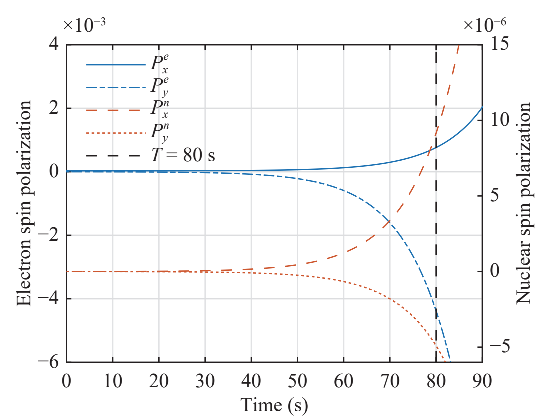

Fig. 2. The polarization trajectory $P_{x/y}^{e/n}$ under the control input $u(t)$. Eventually $P_{x/y}^{e/n}$ achieve $P_{x/y,T}^{e/n}$ with the transition time $T = 80$ s.

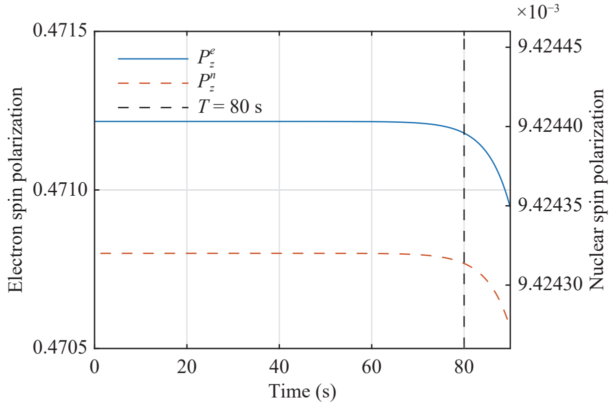

Fig. 3. The longitudinal spin polarization $P_z^{e/n}$ varies slightly with the variation of input $u(t)$, which implies that the fluctuation of the equilibrium point is inevitable when any control measure is implemented on SERFCM system.

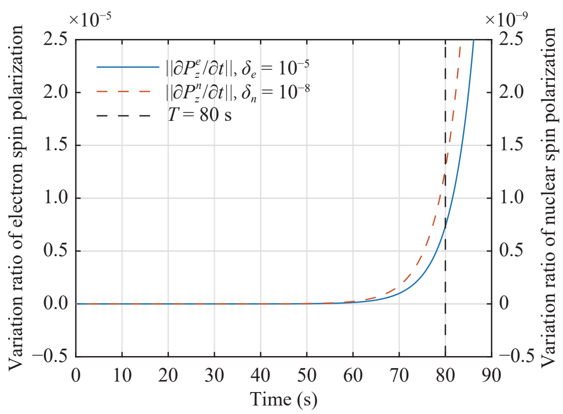

Fig. 4. $||\partial P_z^{e/n}/\partial t||$ with $\sup_t ||\partial P_z^{e/n}/\partial t|| < \delta_{e/n}$ satisfied, which agrees well with the basic assumption that SERFCM system is at PSS during time interval $[0, T]$. This demonstrate that the control input $u(t)$ in Fig. 1 is feasible and applicable.

Remark 6: When the dynamic model of an SERFCM does not satisfy the criterion condition proposed in Theorem 1.

1) One can first analyze the degree of controllability and determine the uncontrollable state components, to find whether there exist some hardware means to change the structural characteristics of the system, for the sake of making the system completely state-controllable. In practical engineering, for SERFCM, its system structural characteristics are basically determined by the core sensitive component, i.e., the vapor cell. Thus, the uncontrollability and instability of the system indicate that the performance parameters of the vapor cell, such as the atomic source combination, the density ratio, and the inner pressure, etc., are unfeasible. Therefore, a vapor cell with feasible performance parameters plays a significant role in SERFCM;

2) If it is impossible to make these uncontrollable state components controllable by physical means, one can then investigate their stability, and determine whether these uncontrollable components are asymptotically stable or at least bounded-input bounded-state (BIBS) stable, to ensure SERFCM can be normally operational.

VI. Conclusion

In this paper, for SERFCM, a nonlinear atomic spin polarization model has been derived from the Bloch equations. In this model, we concern both the variations of transverse as well as longitudinal polarization, and the notion of PSS is introduced according to the real condition. In this case, the model we establish has higher nonlinearity and more difficulty in theoretical analysis, but it is closer to SERFCM in practical engineering with more application value.

For this class of nonlinear systems, we propose and prove a criterion condition of global state controllability, which is the first work to analyze the atomic spin polarization controllability of SERFCM in the related research field. Meanwhile, this criterion condition can also be applied to the controllability analysis for that particular class of nonlinear systems. In addition, we propose a theoretically feasible and reasonable solution of the control input under some physical constraints, which are helpful for the controller design.

We use MATLAB/Simulink simulation software to numerically simulate an K-^{3}He SERFCM model and the simulation results demonstrate the validation of our new controllability determination method.

This paper is the first article of controllability analysis for systems of atomic spin ensembles, whose dynamic behaviors of spin polarization can be described by the Bloch equations. Meanwhile, our conclusions have the potential to be expanded to other systems of precision measuring instruments, which are based on effects of atomic spin and dynamical evolution description by the Bloch equations. Therefore, our theoretical results in this paper are not only pioneering but also of certain universality.

Appendix
The Justification of the Simplification of $\mathcal{H} = \mathcal{H}(x)$

Referred to (8) in Section II, the complete form of matrix $\mathcal{H}(x)$ is expressed as below:

$$\mathcal{H}(x) = \begin{bmatrix} 0 & \gamma_e P_z^e/Q & 0 & -P_z^e \\ -\gamma_e P_z^e/Q & 0 & P_z^e & 0 \\ \gamma_e P_y^e/Q & -\gamma_e P_x^e/Q & -P_y^e & P_x^e \\ 0 & \gamma_n P_z^n & 0 & -P_z^n \\ -\gamma_n P_z^n & 0 & P_z^n & 0 \\ \gamma_n P_y^n & -\gamma_n P_x^n & -P_y^n & P_x^n \end{bmatrix}. \quad (28)$$

From the aspect of strict theoretical proofs, matrix $\mathcal{H}$ indeed vary with the state vector x. But when concerning the practical SERFCM system normally working at PSS, its control input is usually bounded, and the corresponding variations of the 6 polarization components can be calculated according to the Bloch equations. To evaluate the influence of this control input, we define the relative variational rate of $\mathcal{H}$ as below:

$$V_{\mathcal{H}} \triangleq \max_{i,j} \frac{\Delta \mathcal{H}_{ij}}{\mathcal{H}_{ij}}. \quad (29)$$

For example, let the control input be a step magnetic field with the amplitude of 1 nT and the corresponding variations of the 6 polarization components have the typical values as in Table II.

TABLE II
TYPICAL VALUES OF THE POLARIZATION COMPONENTS AND THEIR VARIATIONS FOR SERFCM BASED ON K-^{3}HE ATOMIC SOURCE AT PSS, UNDER THE CONTROL INPUT OF A STEP MAGNETIC FIELD WITH THE AMPLITUDE OF 1 NT

Term	Typical value	Term	Typical value
γ_e	$2\pi\times 2.8\times 10^{10}$ Hz/T	γ_n	$2\pi\times 3.2\times 10^{7}$ Hz/T
P_x^e	3×10^{-4}	ΔP_x^e	1.15×10^{-5}
P_y^e	-2.32×10^{-3}	ΔP_y^e	-4.56×10^{-6}
P_z^e	0.471	ΔP_z^e	1.64×10^{-9}
P_x^n	7.05×10^{-7}	ΔP_x^e	-2.76×10^{-9}
P_y^n	1.24×10^{-6}	ΔP_y^e	-3.85×10^{-5}
P_z^n	9.4×10^{-3}	ΔP_z^e	3.56×10^{-7}
Q	5.27		

In this example, the relative variational rate $V_{\mathcal{H}} = 0.0038\% \approx 0$, which indicates that the variation of $\mathcal{H}$ is very small and can be ignored. One should note that, when SERFCM system is normally working at PSS, the amplitude of the control input is always less than 1 nT, such that the $V_{\mathcal{H}}$ induced by corresponding variations of $x(t)$ is always less than 0.0038%, and also can be ignored. In this situation, $\mathcal{H}(x)$ can be simplified as a constant matrix $\mathcal{H}$ in this paper.

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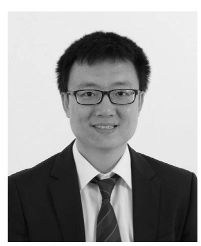

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