Finite-Time Trajectory Tracking Control of a Class of Nonlinear Discrete-Time Systems

Zhuo Wang, *Member, IEEE*, Renquan Lu, *Member, IEEE*, and Hong Wang, *Senior Member, IEEE*

***Abstract*—This paper studies how to control the output of a class of nonlinear discrete-time systems, to completely track any given bounded reference trajectories in finite time. For this problem, we develop two kinds of constructive control methods for the total output case and the partial output case, respectively. For each case, the first kind of methods can design the time instant after which the complete trajectory tracking is accomplished, but cannot guarantee the monotonic decrease of the norm of the tracking error before that time instant; the other kind of methods not only can determine when the output trajectory coincides with the reference trajectory, but also can make the norm of the tracking error decrease monotonically before that time instant. For the partial output case, the proposed control methods can guarantee that the rest part of the system output is bounded for all the time. These control methods are feasible no matter whether the dynamic models of these systems are smooth or nonsmooth. Then, the simulation and experiment results prove the feasibility of the proposed methods.**

***Index Terms*—Bounded reference trajectories, finite-time trajectory tracking, nonlinear discrete-time systems, partial output case, total output case.**

I. INTRODUCTION

IN CONTROL theory and control engineering, trajectory tracking is one of the most important subjects to be extensively studied for linear and nonlinear systems, continuous-time and discrete-time systems, deterministic and uncertain systems, as well as delay and nondelay systems [1]–[5]. Trajectory tracking problems so far been investigated can be divided into two categories: 1) the asymptotic trajectory tracking problems [1], [6]–[9] and 2) the finite-time trajectory tracking problems [3], [10], [11]. Generally speaking, the asymptotic trajectory tracking is necessary to be stable in the sense of Lyapunov [7]–[9]. This means that the norm of the tracking error decreases monotonically as time increases, and the system state/output will converge to the reference trajectory as the time tends to infinity. Besides, it is often assumed that the system model is smooth [1], [12], [13], such as the continuous partial derivatives. People usually prescribe an accuracy (often referred to as the maximum tolerable tracking error), and consider that the trajectory tracking is accomplished when the accuracy requirement is satisfied.

The finite-time trajectory tracking, on the other hand, is quite a different problem. It is concerned with how to design a time instant when the trajectory tracking is accomplished, and how to make the tracking error keep zero from then on. In this sense, the finite-time trajectory tracking is superior to the asymptotic trajectory tracking, since it has higher tracking precision and faster convergence rate, and can determine the exact time instant when the trajectory tracking is accomplished. With the development of science and technology, the requirements on system control accuracy are increasing. Because of the resource limitation or the standard of workmanship, etc., how to make the system state, especially the system output completely track a given reference trajectory in finite time, has become an important control problem. For example, in the areas of shipbuilding and car manufacturing, it has become very common that the industrial robot precisely cuts materials along the preset edge or welds workpieces along the preset welding seam [14], [15]. When launching aircraft, the aircraft is supposed to enter the predesigned trajectory in finite time, and subsequently, flies along this trajectory precisely [16], [17]. In astronomical observation area, the attitude precision and the flying angle along the predesigned trajectory of spacecrafts (such as Hubble Space Telescope), will directly affect the observation of the stars [18], [19]. Therefore, the study and investigation on finite-time trajectory tracking have both theoretical significance and practical value.

Up to now, most of the existing works on finite-time trajectory tracking focus on the control design problem, and some typical control schemes have been developed. Methods studied include the backstepping methods [4], [20], the sliding mode methods [10], [21], and the neural network methods [2], [10], [22]. Of course, the superiority of finite-time trajectory tracking is based on more strict conditions. Consequently, the above control methods have their limitations: they can only deal with some particular systems which have special structures or smooth dynamic characteristics, but cannot control a general class of systems or nonsmooth systems; they can drive the system state/output to track some particular trajectories, but not to track arbitrary bounded trajectories; they can accomplish the complete trajectory tracking

Manuscript received August 30, 2016; revised October 31, 2016; accepted January 21, 2017. Date of publication February 14, 2017; date of current version June 22, 2017. This work was supported by the National Natural Science Foundation of China under Grant 61673041 and Grant 61621004. This paper was recommended by Associate Editor Y.-J. Liu. *(Corresponding author: Zhuo Wang.)*

Z. Wang is with the School of Instrumentation Science and Optoelectronics Engineering, Beihang University, Beijing 100191, China (e-mail: zhuowang@buaa.edu.cn).

R. Lu is with the Institute of Information and Automation, Hangzhou Dianzi University, Hangzhou 310000, China (e-mail: rqlu@hdu.edu.cn).

H. Wang is with the Control Systems Center, University of Manchester, Manchester M60 1QD, U.K. (e-mail: hong.wang@manchester.ac.uk).

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Digital Object Identifier 10.1109/TSMC.2017.2663523

in finite time, but cannot determine the exact time instant when the state/output trajectory begins to coincide with the reference trajectory; most of them are concerned with continuous-time systems, which are not convenient for computer modeling. In view of this situation, further researches need to be conducted, and we have published an article on finite-time trajectory tracking of a class of continuous-time systems [3].

In this paper, we will conduct research on the finite-time output trajectory tracking control of a class of nonlinear discrete-time systems, whose dynamic models can be either smooth or nonsmooth. We will develop control methods in a constructive way, for both the total output case and the partial output case of these systems. Comparing with the previous approaches, our control methods have some advantages: first, they are not limited to particular systems, but can control a general class of nonlinear discrete-time systems; second, they do not require the special structure or the smooth dynamic assumptions of the system model; third, they can be applied to any bounded reference trajectories; and finally, they can arbitrarily design the time instant after which the complete trajectory tracking is accomplished, and some of them even can make the tracking error decrease monotonically before that time instant. These advantages are the main contribution of this paper.

The remainder of this paper is organized as follows. Section II studies how to control the total system output to completely track the given bounded reference trajectory after a predesigned time instant, and how to make the norm of the tracking error decrease monotonically before that time instant. Section III studies how to control a part of the system output to completely track the given bounded reference trajectory in finite time while keeping the other part bounded all the time, and also studies how to make the norm of the tracking error decrease monotonically before the trajectory tracking is accomplished. Section IV demonstrates the feasibility and validity of the proposed control methods by simulation and experiment. In particular, a voice coil motor (VCM) actuated servo gantry system is employed as the experimental subject, where all the data are obtained from the real control experiment. Finally, Section V draws the conclusions of this paper.

II. Finite-Time Total Output Trajectory Tracking Control

Consider the following nonlinear discrete-time system:

$$y(k+1) = f(y(k)) + B(y(k))u(k) \tag{1}$$

where $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^n$ with $m \le n$, are the input and the output, respectively; integer $k \ge 0$ is the tracking instant. Suppose that the mathematical expressions of $f(y(k))$ and $B(y(k))$ are both precisely known; nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in $y(k)$, and $f(0) = 0$; $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is continuous in $y(k)$.

In this section, for the case of $m = n$, we study how to control the total output of system (1) to completely track any given bounded trajectory in finite time. Define the set of the reference trajectories

$$S_1 \triangleq \left\{ g(k) | k \ge 0,\ g(k) \in \mathbb{R}^n, \|g(k)\| \le \alpha < \infty \right\} \tag{2}$$

$g(t)$ is continuous in $t \in \mathbb{R}^+$ and $g(k)$ is the discrete value of $g(t)$ at $t = kT$, where $T > 0$ is the sampling period. In this way, $0 < \alpha < \infty$ is the boundary of any $g(k) \in S_1$. Moreover, we suppose that each given $g(k)$ is known for all $k \ge 0$.

Assumption 1: For system (1), assume that $\forall y(k) \in \mathbb{R}^n$, $\mathrm{Rank}[B(y(k))] = n$.

Assumption 1 is made to ensure that we can arbitrarily control the output $y(k)$ of system (1). To facilitate discussion, we suppose that $\|y(0)\| \le \alpha$ for any given $y(0)$; otherwise, we may first control the output to satisfy this initial condition, and then discuss the trajectory tracking problem.

Theorem 1: Suppose that system (1) satisfies Assumption 1. Then, for any given $y(0)$ and $g(k) \in S_1$, the output $y(k)$ of system (1) can completely track $g(k)$ in finite time.

Proof: If $y(0) = g(0)$, design the control input as

$$u(k) = B^{-1}(y(k))\left[g(k+1) - f(y(k))\right] \quad (k \ge 0). \tag{3}$$

From (1) and (3), we can see that $y(k) = g(k)$ $(k \ge 0)$. Thus, the system output $y(k)$ completely tracks $g(k)$ instantaneously by using control input (3).

If $y(0) \ne g(0)$, predesign a time instant $M \ge 1$ when the complete trajectory tracking is accomplished. Then, we let

$$u(k) = \begin{cases} B^{-1}(y(k))\left[\frac{k+1}{M} g(k+1) - f(y(k))\right] \\ (0 \le k \le M-1) \\ B^{-1}(y(k))\left[g(k+1) - f(y(k))\right] \\ (k \ge M). \end{cases} \tag{4}$$

From (1) and (4), when $k = M - 1$, $y(k+1) = y(M) = [(k+1)/M]g(k+1) = g(M)$; and when $k \ge M$, $y(k+1) = g(k+1)$. Therefore, $y(k)$ completely tracks $g(k)$ from $k = M$ on by using control input (4).

In summary, the output $y(k)$ of system (1) can completely track $g(k) \in S_1$ in finite time. ∎

Note that (3) and (4) are ideal controllers under the assumption that the nonlinear function $f(y(k))$ is precisely known. The following control methods proposed in this paper are also based on this assumption.

Although M can be arbitrarily predesigned, in practical application, it is better to choose M according to the initial tracking error. If $\|y(0) - g(0)\|$ is large, we should choose a large M, such that $y(k)$ will not vary its value dramatically in the initial tracking stage, which might cause problems; otherwise, we may choose a small M. Besides, if $y(0) \ne g(0)$, control input (4) can determine when $y(k)$ begins to coincide with $g(k)$ by choosing proper integer M, which thus is a highlight of our control method.

From the above proof, we can see that when $y(0) = g(0)$, $\|y(k)\| = \|g(k)\| \le \alpha$ for all $k \ge 0$. When $y(0) \ne g(0)$, $\|y(k)\| \le (k/M)\alpha$ for $1 \le k \le M$, and $\|y(k)\| \le \alpha$, for $k \ge M+1$. Since we have supposed that $\|y(0)\| \le \alpha$ for any given $y(0)$, then $\|y(k)\| \le \alpha$ for all $k \ge 0$.

In the above discussion, though not specifically mentioned, the origin of system (1) is usually supposed to be asymptotically stable. Otherwise, according to Assumption 1, system (1) is stabilizable, we can stabilize it first and then study the tracking problem. Note that whether control inputs (3) and (4) are bounded, is not discussed. But, the control input should always be constrained in engineering applications. Suppose

$$\sup_{y(k)\in\mathbb{R}^n}\left\{\left\|B^{-1}(y(k))\right\|\right\} = M_b < \infty. \text{ Then if } y(0) = g(0)$$

from (2) and (3), the control input should satisfy

$$\begin{aligned}\|u(k)\| &\le \left\|B^{-1}(y(k))\right\|\big[\|g(k+1)\| + \|f(y(k))\|\big]\\ &\le M_b\big[\alpha + \|f(y(k))\|\big].\end{aligned}$$

For system (1), when the origin is asymptotically stable, we have $\forall k \ge 0$, $\|f(y(k))\| < \|y(k)\|$. Thus

$$\|u(k)\| < M_b\big[\alpha + \|y(k)\|\big] \le 2\alpha M_b < \infty$$

which indicates the boundary of control input (3) for finite-time trajectory tracking. For the case of $y(0) \ne g(0)$, we can still get the same result, which is omitted here.

It is worth mentioning that the above control method cannot guarantee the monotonic decrease of the tracking error for all $0 \le k \le (M-1)$. As a consequence, this may cause troubles in some particular situations. To overcome this problem, we give an additional assumption below.

Assumption 2: For system (1), $\forall y(k), \bar{y}(k) \in \mathbb{R}^n$, assume that $f(\cdot)$ satisfies $\|f(y(k)) - f(\bar{y}(k))\| \le L_f\|y(k) - \bar{y}(k)\|$, where $0 < L_f < 1$ is a constant number.

Assumption 2 is made to ensure that $\|y(k) - g(k)\|$ decreases monotonically before $y(k)$ begins to coincide with $g(k)$. Before continuing, we preset a constant number $0 < \varepsilon < \infty$ that can be designed to adjust the time instant when the complete trajectory tracking is accomplished.

Theorem 2: Suppose that system (1) satisfies Assumptions 1 and 2. Then, for any given $y(0)$ and $g(k) \in S_1$, the output $y(k)$ of system (1) can completely track $g(k)$ in finite time. In addition, for given constant number $\varepsilon > 0$, when $\|y(0) - g(0)\| > \varepsilon$, there must exist a control input which can make $\|y(k) - g(k)\|$ decrease monotonically before the complete trajectory tracking is accomplished.

Proof: For given constant number $\varepsilon > 0$ and the case of $\|y(0) - g(0)\| \le \varepsilon$, if Assumption 1 is satisfied, we can still use control input (3), which easily leads to $y(k) = g(k)$ $(k \ge 1)$. In particular, when $y(0) = g(0)$, the system output $y(k)$ completely tracks $g(k)$ instantaneously.

When $\|y(0) - g(0)\| > \varepsilon$, let

$$\eta = \log_{L_f}\frac{\varepsilon}{\|y(0) - g(0)\|}, \quad M_\eta = [\eta] + 1 \tag{5}$$

where $[\eta]$ denotes the largest integer less than or equal to η. Then, we design the control input as

$$u(k) = \begin{cases} B^{-1}(y(k))\big[g(k+1) - f(g(k))\big] \\ \big(0 \le k \le M_\eta - 1\big) \\ B^{-1}(y(k))\big[g(k+1) - f(y(k))\big] \\ \big(k \ge M_\eta\big). \end{cases} \tag{6}$$

From (1) and (6), for $0 \le k \le (M_\eta - 1)$, the system output is

$$y(k+1) = g(k+1) + \big[f(y(k)) - f(g(k))\big].$$

By repeatedly using the inequality in Assumption 2, we have

$$\begin{aligned}\|y(k+1) - g(k+1)\| &\le L_f\|y(k) - g(k)\|\\ &\ \ \vdots\\ &\le L_f^{k+1}\|y(0) - g(0)\|.\end{aligned} \tag{7}$$

Since $0 < L_f < 1$, $\|y(k+1) - g(k+1)\| < \|y(k) - g(k)\|$. Therefore, $\|y(k) - g(k)\|$ decreases monotonically for each $0 \le k \le (M_\eta - 1)$. Then, it is reasonable to suppose that

$$\big\|y\big(M_\eta\big) - g\big(M_\eta\big)\big\| \le L_f^{M_\eta}\|y(0) - g(0)\| \le \varepsilon.$$

Let $L_f^\eta\|y(0) - g(0)\| = \varepsilon$. Then, $\eta = \log_{L_f}(\varepsilon/[(\|y(0) - g(0)\|)]$. Let $M_\eta = [\eta] + 1$, and then we can get (5).

From (1) and (6), when $k \ge (M_\eta + 1)$, $y(k) = g(k)$. But, it is not certain whether $y(M_\eta) = g(M_\eta)$. So, we can say with certainty that $y(k)$ completely tracks $g(k)$ after $k = M_\eta$.

In summary, the output $y(k)$ of system (1) can completely track any given $g(k) \in S_1$ in finite time. ■

It is the same as we discussed after the proof of Theorem 1, that control input (6) should also be bounded for bounded reference trajectory. Owing to the limitation of space, we would not go into details here.

The contribution of control method (6) is that it not only can determine the time instant $k = M_\eta$ after which $y(k)$ completely tracks $g(k)$, but also can make $\|y(k) - g(k)\|$ decrease monotonically when $0 \le k \le (M_\eta - 1)$. Here, the constant number ε plays a role as the threshold in determining the control switching point. When $\|y(0) - g(0)\| > \varepsilon$, since $0 < L_f < 1$, $\eta > 0$. If we want $y(k)$ to completely track $g(k)$ in a shorter period of time, we should choose a larger ε in advance; otherwise, we should choose a smaller ε beforehand.

Remark 1: In practice, interference is often inevitable. If system (1) is interfered as

$$y(k+1) = f(y(k)) + B(y(k))u(k) + v(k) \quad (k \ge 0) \tag{8}$$

and if the interference $v(k) \in \mathbb{R}^n$ can be precisely identified in a limited period of time, it can be considered known. In this circumstance, control input (3) is rewritten as

$$u(k) = B^{-1}(y(k))\big[g(k+1) - f(y(k)) - v(k)\big]. \tag{9}$$

Control inputs (4) and (6) can be dealt with similarly, and our control methods are still feasible for finite-time trajectory tracking. Otherwise, if $v(k)$ cannot be precisely identified or is random, these methods cannot accomplish finite-time complete trajectory tracking.

Moreover, from the proofs of Theorems 1 and 2, we can see that system (1) is not required to be smooth, but obviously the proposed control methods are still feasible when the system is smooth. This obviously is another highlight of these methods.

III. Finite-Time Partial Output Trajectory Tracking Control

In practice, the case of $m < n$ is much more common than the case of $m = n$. When $m < n$, if $B(y(k)) \in \mathbb{R}^{n\times m}$ has full column rank, then system (1) can be transformed into the following form through nonsingular transformation:

$$\begin{cases} y_{\mathrm{I}}(k+1) = f_{\mathrm{I}}(y(k)) \\ y_{\mathrm{II}}(k+1) = f_{\mathrm{II}}(y(k)) + G(y(k))u(k) \end{cases} \quad (k \geq 0) \tag{10}$$

where $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^n$ are the input and the output of system (10), respectively. In (10), we denote

$$\begin{aligned} y_{\mathrm{I}}(k) &= \left[y_1(k), y_2(k), \ldots, y_{n-m}(k)\right]^T \\ y_{\mathrm{II}}(k) &= \left[y_{n-m+1}(k), y_{n-m+2}(k), \ldots, y_n(k)\right]^T \\ y(k) &= \left[y_{\mathrm{I}}^T(k), y_{\mathrm{II}}^T(k)\right]^T \\ f_{\mathrm{I}}(y(k)) &= \left[f_1(y(k)), \ldots, f_{n-m}(y(k))\right]^T \\ f_{\mathrm{II}}(y(k)) &= \left[f_{n-m+1}(y(k)), \ldots, f_n(y(k))\right]^T \end{aligned} \tag{11}$$

where $f_{\mathrm{I}} : \mathbb{R}^n \to \mathbb{R}^{n-m}$, $f_{\mathrm{II}} : \mathbb{R}^n \to \mathbb{R}^m$ with $f_{\mathrm{I}}(0) = 0$ and $f_{\mathrm{II}}(0) = 0$; $f_i(y(k))$ $(1 \leq i \leq n)$ are all piecewise continuous in $y(k)$; $G : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is continuous in $y(k)$. Similar to system (1), we suppose that the mathematical expressions of $f_{\mathrm{I}}(y(k))$, $f_{\mathrm{II}}(y(k))$, and $G(y(k))$ are all precisely known.

In this section, we will study how to control the partial output $y_{\mathrm{II}}(k)$ to completely track any given bounded trajectory in finite time, while keeping the other partial output $y_{\mathrm{I}}(k)$ bounded. Define the set of the reference trajectories

$$S_2 \triangleq \left\{h(k) | k \geq 0,\ h(k) \in \mathbb{R}^m, \|h(k)\| \leq \beta < \infty\right\}. \tag{12}$$

$h(t)$ is continuous with respect to $t \in \mathbb{R}^+$ and $h(k)$ is the discrete value of $h(t)$ at $t = kT$. In this way, $0 < \beta < \infty$ is the boundary of any $h(k) \in S_2$. Moreover, we suppose that each given $h(k) \in S_2$ is known for all $k \geq 0$.

Assumption 3: For system (10), assume that $\forall y(k) \in \mathbb{R}^n$:

1) $\|f_{\mathrm{I}}(y(k))\| \leq L_1\|y_{\mathrm{I}}(k)\| + L_2\|y_{\mathrm{II}}(k)\|$, where $0 < L_1 < 1$ and $0 \leq L_2 < \infty$ are constant numbers;
2) $\mathrm{Rank}[G(y(k))] = m$.

1) of Assumption 3 is made to ensure the boundedness of $y_{\mathrm{I}}(k)$ and 2) of Assumption 3 is made to ensure that $y_{\mathrm{II}}(k)$ can be arbitrarily controlled. To facilitate discussion, we suppose that $\|y_{\mathrm{II}}(0)\| \leq \beta$ for any given $y_{\mathrm{II}}(0)$; otherwise, we could control $y_{\mathrm{II}}(k)$ to satisfy this condition first.

Theorem 3: Suppose that system (10) satisfies Assumption 3. Then, for any given $y_{\mathrm{I}}(0)$, $y_{\mathrm{II}}(0)$ and $h(k) \in S_2$, the partial output $y_{\mathrm{II}}(k)$ of system (10) can completely track $h(k)$ in finite time. In the mean time, the other partial output $y_{\mathrm{I}}(k)$ is bounded for all $k \geq 0$.

Proof: When $y_{\mathrm{II}}(0) = h(0)$, design the control input as

$$u(k) = G^{-1}(y(k))\left[h(k+1) - f_{\mathrm{II}}(y(k))\right] \quad (k \geq 0). \tag{13}$$

From (10) and (13), $y_{\mathrm{II}}(k) = h(k)$ $(k \geq 1)$. In this way, $y_{\mathrm{II}}(k)$ completely tracks $h(k)$ instantaneously.

When $y_{\mathrm{II}}(0) \neq h(0)$, predesign a time instant $M \geq 1$ when $y_{\mathrm{II}}(k)$ begins to coincide with $h(k)$. Then, we design

$$u(k) = \begin{cases} G^{-1}(y(k))\left[\frac{k+1}{M}h(k+1) - f_{\mathrm{II}}(y(k))\right] \\ (0 \leq k \leq M-1) \\ G^{-1}(y(k))\left[h(k+1) - f_{\mathrm{II}}(y(k))\right] \\ (k \geq M). \end{cases} \tag{14}$$

From (10) and (14), when $k = M-1$, $y_{\mathrm{II}}(k+1) = y_{\mathrm{II}}(M) = h(M)$; and when $k \geq M$, $y_{\mathrm{II}}(k+1) = h(k+1)$. Then, $y_{\mathrm{II}}(k)$ completely tracks $h(k)$ from $k = M$ on.

In summary, the partial output $y_{\mathrm{II}}(k)$ of system (10) can completely track any given $h(k) \in S_2$ in finite time.

Next, we prove the boundedness of $y_{\mathrm{I}}(k)$. Here, we choose the case of $y_{\mathrm{II}}(0) \neq h(0)$ to demonstrate the proof. The proof for the other case is similar, and thus is omitted here.

By repeatedly using the inequality in 1) of Assumption 3

$$\|y_{\mathrm{I}}(k+1)\| \leq L_1^{k+1}\|y_{\mathrm{I}}(0)\| + L_2\sum_{j=0}^{k} L_1^{k-j}\|y_{\mathrm{II}}(j)\|. \tag{15}$$

Since $0 < L_1 < 1$, $\|y_{\mathrm{II}}(0)\| \leq \beta$ and $\|h(k)\| \leq \beta$ for all $k \geq 0$, then for $0 \leq k \leq (M-1)$, we shall have

$$\begin{aligned} &\|y_{\mathrm{I}}(k+1)\| \\ &\leq L_1^{k+1}\|y_{\mathrm{I}}(0)\| + L_2L_1^k\|y_{\mathrm{II}}(0)\| + L_2\sum_{j=1}^{k} L_1^{k-j}\|y_{\mathrm{II}}(j)\| \\ &< \|y_{\mathrm{I}}(0)\| + L_2\beta + L_2\sum_{j=1}^{M-1}\frac{j}{M}\|h(j)\| \\ &\leq \|y_{\mathrm{I}}(0)\| + \left(1 + \frac{M-1}{2}\right)L_2\beta < \infty. \end{aligned}$$

From (15), for $k \geq M$, the partial output $y_{\mathrm{I}}(k)$ satisfies

$$\begin{aligned} &\|y_{\mathrm{I}}(k+1)\| \\ &\leq L_1^{k+1}\|y_{\mathrm{I}}(0)\| + \beta L_2L_1^k + \beta L_2\sum_{j=1}^{M-1} L_1^{M-1-j}\frac{j}{M} \\ &\quad + \beta L_2\sum_{j=M}^{k} L_1^{k-j} \\ &< \|y_{\mathrm{I}}(0)\| + \beta L_2\sum_{j=0}^{\infty} L_1^j \\ &= \|y_{\mathrm{I}}(0)\| + \frac{1}{1-L_1}\beta L_2 < \infty. \end{aligned}$$

In summary, $y_{\mathrm{I}}(k)$ is bounded for all $k \geq 0$. ■

It is similar to what we discussed in Section II, that when $\sup_{y(k)\in\mathbb{R}^n}\{\|G^{-1}(y(k))\|\} = M_g < \infty$, the control inputs are bounded for bounded reference trajectory.

Note that control input (14) cannot guarantee the monotonic decrease of $\|y_{\mathrm{II}}(k) - h(k)\|$ for all $0 \leq k \leq (M-1)$, which is not expected in some practical applications. To overcome this problem, we give an additional assumption below.

Assumption 4: For system (10), assume that $\forall y(k), \bar{y}(k) \in \mathbb{R}^n$, function $f_{\mathrm{II}} : \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$\|f_{\mathrm{II}}(y(k)) - f_{\mathrm{II}}(\bar{y}(k))\| \leq L_{\mathrm{II}}\|y_{\mathrm{II}}(k) - \bar{y}_{\mathrm{II}}(k)\|$$

where $0 < L_{\mathrm{II}} < 1$ is a constant number, and $\bar{y}_{\mathrm{II}}(k) \in \mathbb{R}^m$ is in the same form of $y_{\mathrm{II}}(k)$ in (11).

Assumption 4 is a lower requirement for the system dynamics than Assumption 2, since it does not require $y_{\mathrm{I}}(k)$ to have the same behavior as $y_{\mathrm{II}}(k)$ does in Assumption 2. Assumption 4 is made to ensure that $\|y_{\mathrm{II}}(k) - h(k)\|$ will decrease monotonically before $y_{\mathrm{II}}(k)$ completely tracks $h(k)$. As in Theorem 2, we preset a constant number $0 < \delta < \infty$ that can be designed to adjust the time instant after which the two trajectories begin to coincide with each other.

Theorem 4: Suppose that system (10) satisfies Assumptions 3 and 4. Then, for any given $y_{\rm I}(0)$, $y_{\rm II}(0)$, and $h(k) \in S_2$, the partial output $y_{\rm II}(k)$ can completely track $h(k)$ in finite time. In addition, for given constant number $\delta > 0$, when $\|y_{\rm II}(0) - h(0)\| > \delta$, there must exist a control input which can make $\|y_{\rm II}(k) - h(k)\|$ decrease monotonically before the trajectory tracking is accomplished. In the mean time, the other partial output $y_{\rm I}(k)$ is bounded for all $k \geq 0$.

Proof: For given constant number $\delta > 0$ and the case of $\|y_{\rm II}(0) - h(0)\| \leq \delta$, if 2) of Assumption 3 is satisfied, we can still use control input (13), which leads to $y_{\rm II}(k) = h(k)$ $(k \geq 1)$. In particular, if $y_{\rm II}(0) = h(0)$, the partial output $y_{\rm II}(k)$ can completely track $h(k)$ instantaneously.

For the case of $\|y_{\rm II}(0) - h(0)\| > \delta$, let

$$\theta = \log_{L_{\rm II}} \frac{\delta}{\|y_{\rm II}(0) - h(0)\|}, \quad M_\theta = [\theta] + 1$$
$$H(k) = \left[0^T,\ h^T(k)\right]^T \in \mathbb{R}^n \tag{16}$$

where $[\theta]$ denotes the largest integer less than or equal to θ. Then, we design the control input as

$$u_{(k)} = \begin{cases} G^{-1}(y(k))\left[h(k+1) - f_{\rm II}(H(k))\right] \\ (0 \leq k \leq M_\theta - 1) \\ G^{-1}(y(k))\left[h(k+1) - f_{\rm II}(y(k))\right] \\ (k \geq M_\theta). \end{cases} \tag{17}$$

From (10) and (17), when $0 \leq k \leq (M_\theta - 1)$

$$y_{\rm II}(k+1) = h(k+1) + \left[f_{\rm II}(y(k)) - f_{\rm II}(H(k))\right].$$

By repeatedly using the inequality in Assumption 4, we have

$$\begin{aligned} \|y_{\rm II}(k+1) - h(k+1)\| &\leq L_{\rm II}\|y_{\rm II}(k) - h(k)\| \\ &\vdots \\ &\leq L_{\rm II}^{k+1}\|y_{\rm II}(0) - h(0)\|. \end{aligned} \tag{18}$$

Since $0 < L_{\rm II} < 1$, $\|y_{\rm II}(k+1) - h(k+1)\| < \|y_{\rm II}(k) - h(k)\|$. Consequently, $\|y_{\rm II}(k) - h(k)\|$ decreases monotonically for each $0 \leq k \leq (M_\theta - 1)$.

From the above discussion, it is reasonable to suppose that

$$\|y_{\rm II}(M_\theta) - h(M_\theta)\| \leq L_{\rm II}^{M_\theta}\|y_{\rm II}(0) - h(0)\| \leq \delta.$$

Let $L_{\rm II}^\theta\|y_{\rm II}(0) - h(0)\| = \delta$. Then, $\theta = \log_{L_{\rm II}}[\delta/(\|y_{\rm II}(0) - h(0)\|)]$. Let $M_\theta = [\theta] + 1$, and then we get (16).

From (10) and (17), when $k \geq (M_\theta + 1)$, $y_{\rm II}(k) = h(k)$. But, it is not certain whether $y_{\rm II}(M_\theta) = h(M_\theta)$. Hence, we can say with certainty that $y_{\rm II}(k)$ completely tracks $h(k)$ after time instant $k = M_\theta$.

In summary, $y_{\rm II}(k)$ can completely track $h(k) \in S_2$ in finite time. The rest part is about the boundedness of $y_{\rm I}(k)$, which is similar to that part of the proof of Theorem 3, and thus is omitted here. ■

The contribution of control method (17) is that it not only can determine the time instant $k = M_\theta$ after which $y_{\rm II}(k)$ completely tracks $h(k)$, but also can make $\left\|y_{\rm II}(k) - h(k)\right\|$ decrease monotonically for all $0 \leq k \leq (M_\theta - 1)$. Here, the constant number δ plays a role as the threshold in determining the control switching point. When $\left\|y_{\rm II}(0) - h(0)\right\| > \delta$, since $0 < L_{\rm II} < 1$, $\theta > 0$. If we want $y_{\rm II}(k)$ to completely track $h(k)$ faster, we should choose a larger δ in advance; otherwise, we should choose a smaller δ beforehand.

Remark 2: Similar to what we discussed in Section II, if system (10) is interfered as

$$\begin{cases} y_{\rm I}(k+1) = f_{\rm I}(y(k)) \\ y_{\rm II}(k+1) = f_{\rm II}(y(k)) + G(y(k))u(k) + w(k) \end{cases} \tag{19}$$

and if the interference $w(k) \in \mathbb{R}^m$ can be precisely identified within limited time, the control methods proposed above are still feasible for finite-time trajectory tracking. Take control input (13) for example, it will be rewritten as

$$u(k) = G^{-1}(y(k))\left[h(k+1) - f_{\rm II}(y(k)) - w(k)\right] \tag{20}$$

and control inputs (14) and (17) can be dealt with similarly. Otherwise, if $w(k)$ cannot be precisely identified or is random, our control methods are no longer feasible.

In addition, no matter system (10) is smooth or nonsmooth, the proposed control methods are applicable.

In practical engineering, there is a class of 2-D systems which have the same structure of system (10). They represent the dynamics of an object that moves along the preset track, such as the linear motor systems [1], [23], etc. Their Euler discretized models are in the following form:

$$\begin{cases} y_1(k+1) = y_1(k) + Ty_2(k) \\ y_2(k+1) = f_2(y(k)) + G(y(k))u(k) \end{cases} \quad (k \geq 0) \tag{21}$$

where $u(k) \in \mathbb{R}$, $y(k) = [y_1(k), y_2(k)]^T \in \mathbb{R}^2$ $(m = 1, n = 2)$ are the input and output; $f_2 : \mathbb{R}^2 \to \mathbb{R}$ is piecewise continuous in $y(k)$ and $f_2(0) = 0$; $G(y(k))$ is continuous in $y(k)$.

In system (21), $y_1(k)$ and $y_2(k)$ represent the position and velocity of the moving object, respectively; and $T > 0$ is the sampling period. We will study how to control $y_2(k)$ to make $y_1(k)$ completely track a given bounded trajectory in finite time. Define the set of reference trajectories

$$S_3 \triangleq \{q(k) | k \geq 0,\ q(k) \in \mathbb{R},\ |q(k)| \leq \zeta < \infty\}. \tag{22}$$

$|\cdot|$ denotes the absolute value. $q(t)$ is continuous with respect to $t \in \mathbb{R}^+$, and $q(k)$ is the discrete value of $q(t)$ at $t = kT$. Then, $0 < \zeta < \infty$ is the boundary of any $q(k) \in S_3$. We suppose that each given $q(k) \in S_3$ is known for all $k \geq 0$.

Theorem 5: Suppose that in system (21), $G(y(k)) \neq 0$ for all $y(k) \in \mathbb{R}^2$. Then, for any given $y_1(0)$, $y_2(0)$, $T > 0$, and $q(k) \in S_3$, the partial output $y_1(k)$ of system (21) can completely track $q(k)$ in finite time. In the mean time, the other partial output $y_2(k)$ is bounded for all $k \geq 0$.

Proof: Predesign a time instant $M \geq 3$ from which on $y_1(k)$ begins to completely track $q(k)$, and let

$$\gamma = \frac{2\left[q(M) - y_1(0) - Ty_2(0)\right]}{T^2(M-1)(M-2)}. \tag{23}$$

Then, we design the control input as

$$u(k) = \begin{cases} G^{-1}(y(k))\left[\gamma kT - f_2(y(k))\right] \\ (0 \leq k \leq M - 1) \\ G^{-1}(y(k))\left[\frac{q(k+2)-q(k+1)}{T} - f_2(y(k))\right] \\ (k \geq M). \end{cases} \tag{24}$$

From (21) and (24), for $0 \le k \le (M-1)$, $y_2(k+1) = \gamma kT$. From (21) and (23), we can obtain

$$\begin{aligned} y_1(M) &= y_1(M-1) + Ty_2(M-1) \\ &= y_1(0) + Ty_2(0) + \gamma T^2 \sum_{j=0}^{M-2} j \\ &= y_1(0) + Ty_2(0) + \frac{1}{2}\gamma T^2 (M-1)(M-2) \\ &= q(M). \end{aligned} \tag{25}$$

Substitute (24) into (21), for $k \ge M$, we shall have

$$\begin{aligned} y_2(k+1) &= \frac{1}{T}\big[q(k+2) - q(k+1)\big] \\ y_1(k+1) &= q(k+1) + \big[y_1(k) - q(k)\big]. \end{aligned} \tag{26}$$

Observe (25) and (26), $\forall k \ge M$, $y_1(k) = q(k)$. Then, $y_1(k)$ can completely track any given $q(k) \in S_3$ from $k = M$ on.

On the other hand, we can see that

$$\begin{aligned} &|y_2(k)| \le \max\{|y_2(0)|, |\gamma| T(M-1)\}, \quad (0 \le k \le M) \\ &|y_2(k)| \le \frac{1}{T}\big[|q(k+1)| + |q(k)|\big] \le \frac{2}{T}\zeta, \quad (k \ge M+1). \end{aligned}$$

In summary, $y_2(k)$ is bounded for all $k \ge 0$. ■

IV. Simulation and Experiment Study

In this section, we will demonstrate the research results of this paper by simulation and experiment.

A. Simulation Example of Finite-Time Total Output Trajectory Tracking Control

For the total output case, where $m = n$, we present the following simulation example to illustrate Theorems 1 and 2.

Example 1: Consider a nonlinear discrete-time system in the form of (1), where

$$\begin{aligned} f(y(k)) &= \begin{bmatrix} \frac{1}{3}y_2(k) \\ \frac{1}{3}\mathrm{sat}(y_1(k) + y_2(k)) \end{bmatrix} \\ B(y(k)) &= \begin{bmatrix} 2 + \cos(y_1(k)) & 0 \\ -1 & 3 \end{bmatrix}. \end{aligned} \tag{27}$$

In this system, $m = n = 2$ and $\mathrm{sat}(\cdot)$ represents the saturation function. Both $f(y(k))$ and $B(y(k))$ are continuous with respect to $y(k)$ and $f(0) = 0$. In addition, $\forall y(k) \in \mathbb{R}^2$, $\mathrm{Rank}[B(y(k))] = 2$. Thus, system (27) satisfies the description of model (1) and Assumption 1.

Based on the fact that $\forall a, b \in \mathbb{R}$, $|\mathrm{sat}(a) - \mathrm{sat}(b)| \le |a - b|$, $\forall y, \bar{y} \in \mathbb{R}^2$, we shall have

$$\begin{aligned} &\|f(y) - f(\bar{y})\|^2 \\ &\le \frac{1}{9}\Big[(y_1 - \bar{y}_1)^2 + 2(y_1 - \bar{y}_1)(y_2 - \bar{y}_2) + 2(y_2 - \bar{y}_2)^2\Big]. \end{aligned}$$

Then, $\forall y, \bar{y} \in \mathbb{R}^2$, $\|f(y) - f(\bar{y})\| \le 0.5393\|y - \bar{y}\|$. In the above, we used the property that $y^T P y \le \lambda_{\max}(P) y^T y$ for all $y \in \mathbb{R}^n$, where $\lambda_{\max}(P)$ is the maximum eigenvalue of matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. In this case, Assumption 2 is satisfied with $L_f = 0.5393$.

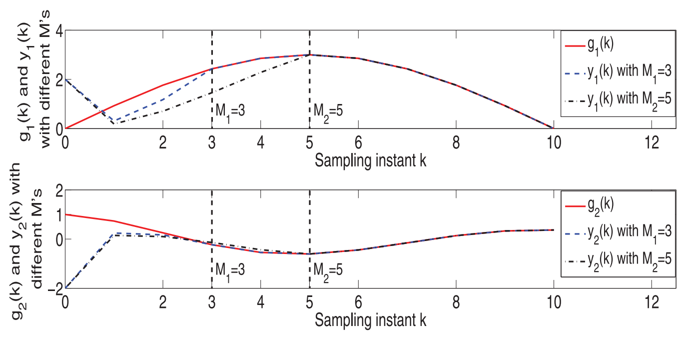

Fig. 1. Simulation of Theorem 1 with $M_1 = 3$ and $M_2 = 5$.

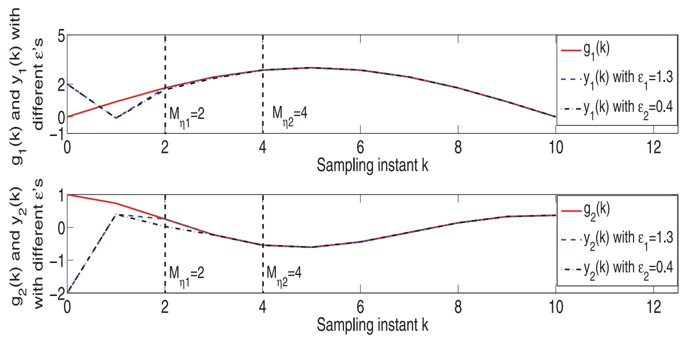

Fig. 2. Simulation of Theorem 2 with $\varepsilon_1 = 1.3$, $M_{\eta 1} = 2$, $\varepsilon_2 = 0.4$, and $M_{\eta 2} = 4$.

In this example, the reference trajectory is given as

$$g(k) = \begin{bmatrix} 3\sin(0.1k\pi) \\ \cos(0.2k\pi)\exp(-0.1k) \end{bmatrix} \quad (k \ge 0). \tag{28}$$

Hence, $g(0) = [0, 1]^T$ and $\|g(k)\| \le \sqrt{10}$ for all $k \ge 0$, which satisfies the definition of S_1 in (2).

For both Theorems 1 and 2, in order to explain the effects of parameters M and ε in the corresponding control algorithms, we will use the same initial value $y(0) = [2, -2]^T$, such that $y(0) \ne g(0)$. For Theorem 1, we set $M_1 = 3$ and $M_2 = 5$, to illustrate that one can arbitrarily design the time instant after which $y(k)$ completely tracks $g(k)$.

With (1), (4), (27), and (28), we use MATLAB program to simulate the performances of trajectory tracking. The simulation results shown in Fig. 1 accord with Theorem 1, that $y(k)$ can completely track $g(k)$ from $k = 3$ ($k = 5$) on. As we discussed after the proof of Theorem 1, control law (4) cannot guarantee the monotonic decrease of the norm of the tracking error for all $0 \le k \le 2$ ($0 \le k \le 4$).

For Theorem 2, arbitrarily design $\varepsilon_1 = 1.3$ and $\varepsilon_2 = 0.4$, which are both less than $\|y(0) - g(0)\| = 3.6056$. From (5), with $L_f = 0.5393$, we have $\eta_1 = 1.6521$ and $\eta_2 = 3.5609$. As a result, $[\eta_1] = 1$, $[\eta_2] = 3$, $M_{\eta 1} = 2$, and $M_{\eta 2} = 4$.

We then use MATLAB program to simulate the performances of trajectory tracking according to (1), (6), (27), and (28). The simulation results shown in Fig. 2 accord with Theorem 2 that control input (6) not only can make $y(k)$ completely track $g(k)$ after the time instant $M_{\eta 1} = 2$ ($M_{\eta 2} = 4$), but also can make $\|y(k) - g(k)\|$ decrease monotonically for all $0 \le k \le 1$ ($0 \le k \le 3$).

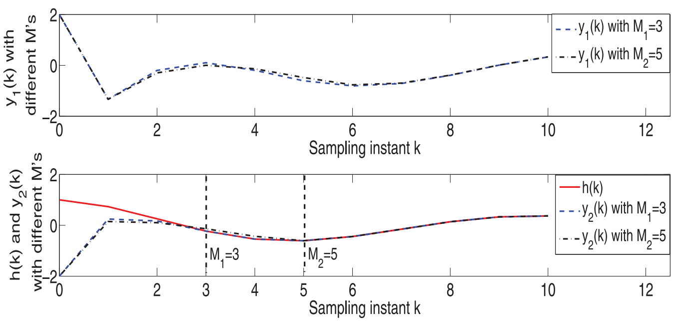

Fig. 3. Simulation of Theorem 3 with $M_1 = 3$ and $M_2 = 5$.

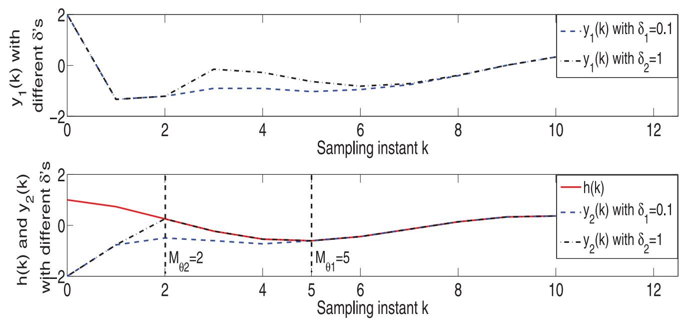

Fig. 4. Simulation of Theorem 4 with $\delta_1 = 0.1$, $M_{\theta 1} = 5$, $\delta_2 = 1$, and $M_{\theta 2} = 2$.

Besides, as we discussed after the proof of Theorem 2, that when $\|y(0) - g(0)\| > \varepsilon$, a larger ε will bring a smaller M_η and a faster convergence speed.

B. Simulation and Experiment Examples of Finite-Time Partial Output Trajectory Tracking Control

For the partial output case, where $m < n$, we will give a simulation example and an experiment example to illustrate the theorems in Section III.

Example 2: For Theorems 3 and 4, consider a nonlinear discrete-time system in the form of (10), where

$$f_{\rm I}(y(k)) = \frac{1}{3}y_1(k) + y_2(k), \quad f_{\rm II}(y(k)) = \mathrm{sat}(0.5y_2(k))$$
$$G(y(k)) = 2 + \sin(y_1(k) - y_2(k)). \tag{29}$$

Here, $m = 1$ and $n = 2$; $f_{\rm I}(y(k))$, $f_{\rm II}(y(k))$, and $G(y(k))$ are all continuous with respect to $y(k) = [y_1(k), y_2(k)]^T \in \mathbb{R}^2$; $f_{\rm I}(0) = 0$ and $f_{\rm II}(0) = 0$. Therefore, system (29) satisfies the description of model (10). Besides, $\forall y, \bar{y} \in \mathbb{R}^2$, we have

$$\|f_{\rm I}(y)\| \le \frac{1}{3}|y_1| + |y_2|, \quad \mathrm{Rank}\big[G(y)\big] = 1$$
$$\|f_{\rm II}(y) - f_{\rm II}(\bar{y})\| = |\mathrm{sat}(0.5y_2) - \mathrm{sat}(0.5\bar{y}_2)| \le 0.5|y_2 - \bar{y}_2|.$$

Then, Assumptions 3 and 4 are both satisfied, with $L_1 = (1/3)$ and $L_{\rm II} = 0.5$. The reference trajectory is given as

$$h(k) = \cos(0.2k\pi)\exp(-0.1k) \quad (k \ge 0). \tag{30}$$

Then, $h(0) = 1$ and $|h(k)| \le 1$ for all $k \ge 0$.

For both Theorems 3 and 4, in order to explain the effects of M and δ in the corresponding control algorithms, we will use the same initial value $y(0) = [2, -2]^T$, such that $y_2(0) \ne h(0)$. For Theorem 3, we set $M_1 = 3$ and $M_2 = 5$ to illustrate that one can arbitrarily design the time instant from which on $y_2(k)$ starts to completely track $h(k)$ by choosing proper M.

Based on (10), (14), (29), and (30), we use MATLAB program to simulate the performances of trajectory tracking. Fig. 3 shows the simulation results, which accord with Theorem 3, that $y_2(k)$ can completely track $h(k)$ from $k = 3$ ($k = 5$) on, and $y_1(k)$ is bounded for all $k \ge 0$.

For Theorem 4, arbitrarily design $\delta_1 = 0.1$ and $\delta_2 = 1$, which are both less than $|y_2(0) - h(0)| = 3$. From (16), with $L_{\rm II} = 0.5$, we have $\theta_1 = 4.907$ and $\theta_2 = 1.585$. As a result, $[\theta_1] = 4$, $[\theta_2] = 1$, $M_{\theta 1} = 5$, and $M_{\theta 2} = 2$. We then use MATLAB program to simulate the performances of trajectory tracking according to (10), (17), (29), and (30).

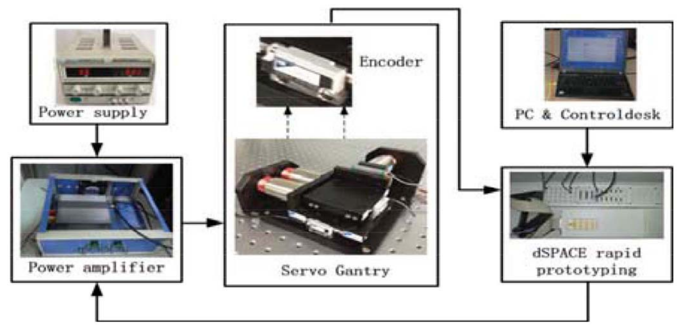

Fig. 5. VCM actuated servo gantry system.

The simulation results shown in Fig. 4 accord with Theorem 4, that control input (17) not only can design the time instant $M_{\theta 1} = 5$ ($M_{\theta 2} = 2$) after which $y_2(k)$ completely tracks $h(k)$, but also can make $|y_2(k) - h(k)|$ decrease monotonically for all $0 \le k \le 4$ ($0 \le k \le 1$). In the mean time, $y_1(k)$ is bounded for all $k \ge 0$. Besides, as we discussed after the proof of Theorem 4, that when $|y_2(0) - h(0)| > \delta$, a larger δ will bring a smaller M_θ and a faster convergence rate.

For Theorem 5, we will use an experiment example to illustrate the effectiveness and practicability of control method (24). The experimental subject is the servo gantry in a VCM actuated servo gantry system, which is depicted in Fig. 5. The manufacturer of this device is *H2W Technologies, Inc.* and the product code is NCC10-15-023-1X.

Example 3: The dynamic model of this servo gantry is

$$\begin{cases} y_1(k+1) = y_1(k) + Ty_2(k) \\ y_2(k+1) = y_2(k) - \frac{KT}{M_{\rm vcm}}y_1(k) - \frac{T}{M_{\rm vcm}}F_f(k) \\ \qquad\qquad + \frac{TK_F K_u}{M_{\rm vcm}}u(k) - \frac{T}{M_{\rm vcm}}F_{\rm int}(k). \end{cases} \tag{31}$$

In system (31), $m = 1$ and $n = 2$; $y_1(k)$ (in μm) and $y_2(k)$ (in mm/s) represent the position and velocity of the VCM, respectively; $T = 0.1$ ms is the sampling period; $u(k)$ (in V) is the control voltage; $F_f(k)$ (in N) and $F_{\rm int}(k)$ (in N) are the frictional force and interference force, respectively, where

$$F_f(k) = \Big\{F_c + (F_s - F_c)\exp\Big[-(y_2(k)/V_s)^2\Big] + F_v y_2(k)\Big\} \times \mathrm{sgn}(y_2(k))$$
$$F_{\rm int}(k) = 0.0073\sin(0.1Tk\pi). \tag{32}$$

Obviously, $f_2(y(k)) = y_2(k) - (KT/M_{\rm vcm})y_1(k) - (T/M_{\rm vcm})F_f(k)$ is piecewise continuous in $y(k) =$

TABLE I
PARAMETERS OF THE VCM ACTUATED SERVO GANTRY SYSTEM

Contents	Units	Values
K	N·m^{-1}	96.51
K_F	N·A^{-1}	32.36
M_{vcm}	Kg	0.82
K_u	A·V^{-1}	0.5
F_c	N	0.25
F_s	N	0.3
V_s	m/s	0.001
F_v	N·s·m^{-1}	94.63

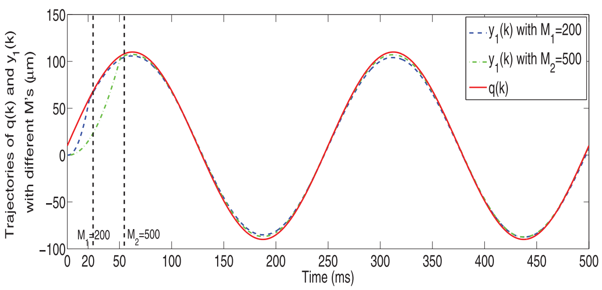

Fig. 6. Position trajectories with $M_1 = 200$ and $M_2 = 500$.

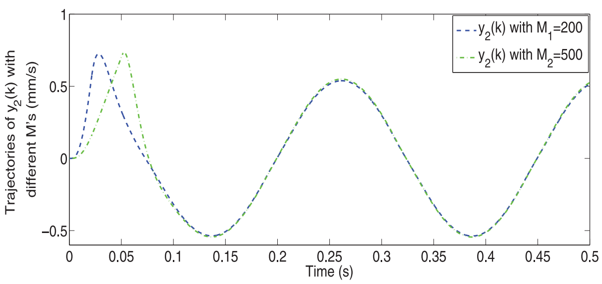

Fig. 7. Velocity trajectories with $M_1 = 200$ and $M_2 = 500$.

$[y_1(k), y_2(k)]^T$, $f_2(0) = 0$; and $G(y(k)) = (TK_F K_u/M_{\text{vcm}})$ is a constant. Therefore, system (31) satisfies the description of model (21).

The corresponding parameters are introduced in Table I.

As we discussed in Remark 2, since the interference $w(k) = -(T/M_{\text{vcm}})F_{\text{int}}(k)$ (in mm/s) is clearly identified, control input (24) can still be used just by adjusting it to

$$u(k) = \begin{cases} G^{-1}(y(k))\left[\gamma kT - f_2(y(k)) - w(k)\right] \\ (0 \leq k \leq M-1) \\ G^{-1}(y(k))\left[\frac{q(k+2)-q(k+1)}{T} - f_2(y(k)) - w(k)\right] \\ (k \geq M). \end{cases}$$

In this experiment, the reference trajectory is given as

$$q(k) = 10 + 100\sin(8Tk\pi) \quad (k \geq 0). \tag{33}$$

Then, $q(0) = 10$ and $|q(k)| \leq 110$ (in μm) for all $k \geq 0$, which satisfies the definition of S_3 in (22).

We set $y(0) = [0, 0]^T$ and choose $M_1 = 200$, $M_2 = 500$ to illustrate the effect of parameter M. Figs. 6 and 7 show the experiment results, which basically accord with Theorem 5, that one can arbitrarily design the time instant $k = M$ after which $y_1(k)$ completely tracks $q(k)$, while keeping $y_2(k)$ bounded for all $k \geq 0$.

Note that in Fig. 6 the position $y_1(k)$ does not track $q(k)$ very well at the peaks and valleys. This is because the servo gantry changes the moving direction at these places, but the inertia and frictional force will impede the proper movements of it. Besides, model (31) actually has modeling errors that certainly influence the control precision.

V. CONCLUSION

The proposed control methods in this paper are applied to make the system output (or, part of it) completely track the reference trajectory in finite time (while keeping the other part of the system output bounded). One highlight is that they are feasible no matter whether the dynamic models of the controlled systems are smooth or nonsmooth.

The control methods proposed in the proofs of Theorems 1, 3, and 5 can arbitrarily design the time instant when the (partial) output trajectory and the reference trajectory begin to coincide with each other, but they cannot guarantee the monotonic decrease of the norm of the tracking error before that time instant. The control methods proposed in the proofs of Theorems 2 and 4 not only can determine the time instant after which the complete trajectory tracking is accomplished, but also can make the norm of the tracking error decrease monotonically before that time instant. On the other hand, for the partial output case, the control methods proposed in the proofs of Theorems 3–5 can keep the other partial output bounded for all the time. Another highlight of these control methods is that they are designed in a constructive way. The above advantages are the main contribution of this paper.

In the end, the feasibility and validity of these control methods are demonstrated by simulation and practical experiment.

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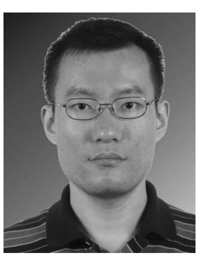

Zhuo Wang (M'15) was born in Handan, China, in 1983. He received the B.E. degree in automation from Beihang University, Beijing, China, in 2006, and the Ph.D. degree in electrical and computer engineering from the University of Illinois at Chicago, Chicago, IL, USA, in 2013.

He was a Post-Doctoral Fellow with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada, from 2013 to 2014. He was a Research Assistant Professor with the Fok Ying Tung Graduate School, Hong Kong University of Science and Technology, Hong Kong, from 2014 to 2015. He was selected for the Thousand Talents Program for Distinguished Young Scholars by the Organization Department of the CPC Central Committee and the 100 Talents Program by Beihang University, in 2015. He is currently a Professor with the School of Instrumentation Science and Optoelectronics Engineering, Beihang University.

Renquan Lu (M'06) was born in Hubei, China, in 1971. He received the Ph.D. degree in automation from Zhejiang University, Hangzhou, China in 2004.

In 2008, he served as a Visiting Professor with the Department of Electrical and Computer Engineering, University of Newcastle, Callaghan, NSW, Australia, for six months. He is currently a Full Professor with the Guangdong Key Laboratory of IOT Information Processing, Guangdong University of Technology, Guangzhou, China. He has published over 60 journal papers in the fields of robust control and networked control systems.

Hong Wang (SM'06) received the B.S. degree from the Huainan University of Mining Engineering, Huainan, China, in 1982, and the M.S. and Ph.D. degrees from the Huazhong University of Science and Technology, Wuhan, China, in 1984 and 1987, respectively.

He was a Research Fellow with Salford University, Salford, U.K., Brunel University, Uxbridge, U.K., and Southampton University, Southampton, U.K. In 1992, he joined the University of Manchester Institute of Science and Technology, Manchester, U.K., in 1992, where he was a Professor of Process Control from 2002 to 2016. He has been with Pacific Northwest National Laboratory, Richland, WA, USA, as a Laboratory Fellow and a Chief Scientist since 2016. His current research interests include stochastic distribution control, fault detection and diagnosis, nonlinear control, and data-based modeling for complex systems. He has published 200 papers and three books in the above areas.

Dr. Wang was an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and currently serves as an Associate Editor for the IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY and the IEEE TRANSACTIONS ON AUTOMATION SCIENCE AND ENGINEERING.